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Sorting with TOPSIS through boundary and characteristic profiles

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ABSTRACT

This paper proposes new versions of the TOPSIS method for Multiple Criteria Ordinal Classification (sorting). We apply features found in the literature to prevent ranking reversals in TOPSIS and their impacts on sorting problems. Thus, TOPSIS-Sort-B is presented as an improved version of TOPSIS-Sort for sorting problems in which boundary profiles should be determined. In addition, we propose a novel TOPSIS-based sorting method, called TOPSIS-Sort-C, that should be used to address problems in which it is more appropriate to determine characteristic profiles. Both methods were applied in a numerical application that assessed the degree of economic freedom of 180 countries and assigned them to five pre-defined ordered classes. The results showed coherence when compared with the ratings already found in the literature and provided by a specialized institution.

1. Introduction

In Multiple Criteria Decision Making/Aiding (MCDM/A), the sorting/classification problematic (Doumpos & Zopounidis, 2002; Hwang & Yoon, 1996; Zopounidis & Doumpos, 2002a) describes problems in which alternatives should be allocated to a set of pre-defined categories/classes in accordance with how a set of criteria has evaluated these alternatives. In the Multiple Criteria Ordinal Classification, commonly referred to as sorting, the pre-defined classes are ordered by preference. In this case, alternatives allocated to the most preferred categories are better than alternatives allocated to the less preferred categories. Furthermore, each alternative should be allocated to only one category (Doumpos & Zopounidis, 2002; Ferreira, Borenstein, Righi, & de Almeida Filho, 2018). On the other hand, in Multiple Criteria Nominal Classification, the categories are not ordered by preference and the alternatives are grouped according to their similarities. In this case, an alternative may be allocated to more than one group or may not be allocated at all (Ferreira et al., 2018) and it is not possible to establish preference relations between alternatives from different groups.

Over the last 20 years, several methods have been developed to solve sorting problems. ELECTRE-TRI and its variations (Almeida-Dias, Figueira, & Roy, 2010, 2012; Fernández, Figueira, Navarro, & Roy, 2017; Micale, La Fata, & La Scalia, 2019; Mousseau & Slowinski, 1998; Ramezanian, 2019) represent examples of sorting approaches that are applied in outranking problems. Adaptations of the PROMETHEE outranking method to address sorting problems have also been proposed (de Silva, 2018; Doumpos & Zopounidis, 2004), where PROMETHEE parameters are inferred by using mathematical programming techniques. In UTADIS (Jacquet-Lagreze, 1995), linear programming is used to obtain an additive function and to sort alternatives into ordered classes. Other examples of MCDM/A sorting approaches include DRSA variations (Greco, Matarazzo, & Slowinski, 2001; Kadziński, Greco, & Słowiński, 2014; Słowinski, Greco, & Matarazzo, 2012), AHP-Sort (Ishizaka, Pearman, & Nemery, 2012), MACBETH-Sort (Ishizaka & Gordon, 2017), and the Additive-Veto sorting approach (Palha, de Almeida, Morais, & Hipel, 2019).

TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) is among the most popular MCDM/A methods. Traditionally, this method is applied to ranking problems, where alternatives are evaluated based on Euclidean distances from an ideal and a nonideal solution. Behzadian, Khanmohammadi Otaghsara, Yazdani, and Ignatius (2012) present a systematic review of TOPSIS applications. This showed that more than 200 papers were published between 2000 and 2012, which illustrates the potential and applicability of the method. In this review, nine application areas were identified, including for instance: Supply Chain Management and Logistics; Design, Engineering and Manufacturing Systems; Business and Marketing Management; and Health, Safety, and Environment Management. Furthermore, Salih, Zaidan, Zaidan, and Ahmed (2019) present a survey of Fuzzy-TOPSIS, which considers 170 articles published between 2007 and 2017. The use of fuzzy sets is interesting in MCDM/A problems since fuzzy numbers can express linguistic evaluations (Ferreira et al., 2018). Proposals on the assessment of linguistic variables and extensions of group decision making with multiple criteria have been proposed lately (Yu, Zhang, Zhong, & Sun, 2017; Zhang, Guo, & Martinez,

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2017). Recent developments on TOPSIS include a solution for the rank reversal problem (de Aires, 2019), a risk TOPSIS methodology for applications in risk-based preventive maintenance (Seiti & Hafezalkotob, 2019), a TOPSIS method based on a fuzzy covering approximation space (Zhang, Zhan, & Yao, 2019), a T2NN-TOPSIS approach for group decision making under a type-2 neutrosophic number (Abdel-Basset, Saleh, Gamal, & Smarandache, 2019), and an interval-valued TOPSIS (Micale et al., 2019) which was applied combined with interval-valued ELECTRE-TRI for a storage location assignment problem. Recently, variations of TOPSIS to address sorting/classification problems have been proposed. Sabokbar, Hosseini, Banaitis, and Banaitiene (2016) presented the TOPSIS-Sort approach. In this method, upper and lower limit profiles are determined for the categories and their closeness coefficients are calculated according to the TOPSIS procedure. Then, the classification of alternatives is made by comparing their closeness coefficients to those of the profiles. Ferreira et al. (2018) proposed the FTOPSIS-Class, a variant of FTOPSIS (Chen, 2000). The method was used to classify investment alternatives according to investors‘ risk-taking profiles (conservative, moderate, bold and aggressive).

Despite the TOPSIS methods mentioned above, the authors consider that there are gaps in the literature, which are discussed in this paper. In the TOPSIS-Sort original routine, alternatives are sorted according to how their closeness coefficients are ranked compared to those of the profiles, which play the role of boundary references for the classes. Therefore, TOPSIS ranking reversal may occur and the allocation of a specific alternative may change if an external alternative is added to the problem. Moreover, for several situations, it is more appropriate to determine characteristic profiles than to work with boundary profiles. Finally, there are still few examples in the literature of numerical applications regarding TOPSIS for sorting problems.

The contributions of this paper are threefold. First, it proposes a variation on the TOPSIS-Sort routine, called TOPSIS-Sort-B, by using solutions from the literature on TOPSIS for the ranking reversal problem. In short, TOPSIS-Sort-B defines the use of only $|class| - 1$ boundary profiles instead of using upper and lower limiting profiles for each class; this includes a step for defining a domain for the criteria; and it adds an interval normalization option, which is more appropriate for situations where criteria have different domains. Secondly, this paper proposes a novel TOPSIS-Sort method, called TOPSIS-Sort-C, where the sorting process is based on using characteristic profiles, determined in a constructive perspective. Thirdly, the methods are applied to sort the degree of economic freedom of 180 countries into five ordered classes. This application demonstrates the use of the new method and the results are discussed.

The rest of the paper is organized as follows. Section 2 presents the traditional TOPSIS and TOPSIS-Sort routines and discusses the classification change problem with an example. In Section 3, TOPSIS-Sort variations for boundary and characteristic profiles are presented, where we propose TOPSIS-Sort-C. In Section 4, a numerical application of TOPSIS-Sort-C is detailed. Finally, conclusions are drawn and suggestions for future studies are made in Section 5.

2. Preliminaries

Before starting the presentation of the procedures, we consider it will be helpful to introduce the reader at this point to some notation that will be used for the MCDM/A methods throughout the paper. Let $A = \{a_1, a_2, a_3, \cdots, a_m\}$ be a set of m alternatives; $G = \{g_1, g_2, g_3, \cdots, g_n\}$ be a set of n criteria; $W = [w_1, w_2, w_3, \cdots, w_n]$ be a vector of weights with n elements, where w_j is the weight of criterion g_j; and $C = \{C_1, C_2, \cdots, C_q\}$ be a set of q pre-defined ordered classes of a sorting problem . Let G^+ and G^- be respectively the subsets of beneficial and cost criteria. Let $a_{i,j}$ be the performance of alternative a_i regarding criterion g_j and let $P_{k,j}$ be the performance of profile P_k regarding criterion g_j.

2.1. TOPSIS and TOSPSIS-sort procedures

Algorithm 1 gives a detailed description of the traditional TOPSIS procedure, proposed in (Hwang & Yoon, 1981) for ranking problems. The alternatives are evaluated according to their distances to ideal and anti-ideal solutions. Therefore, the nearer an alternative is to the ideal solution, the better the evaluation of that alternative. The calculations are made using Euclidean distances (Step 4) and a closeness coefficient is also calculated (Step 5).

Algorithm 1: TOPSIS Routine

Step 1: Normalize the decision matrix $X = [a_{i,j}]_{m\times n}$, finding $R = [r_{i,j}]_{m\times n}$

$$r_{i,j} = \frac{a_{i,j}}{\sqrt{\sum_{i=1}^{m} a_{i,j}^{2}}}$$

Step 2: Calculate the decision matrix $V = [v_{i,j}]_{m\times n}$ normalized by the weights.

$v_{i,j} = w_j r_{i,j}$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$

Where: $\sum_{j=1}^{n} w_j = 1$

Step 3: Determine the ideal and anti-ideal solutions.

$$v^* = [v_1^*, v_2^*, \cdots, v_n^*],\ v_j^* = \begin{cases} \max_i v_{i,j}, & g_j \in G^+ \\ \min_i v_{i,j}, & g_j \in G^- \end{cases}$$

$$v^- = [v_1^-, v_2^-, \cdots, v_n^-],\ v_j^- = \begin{cases} \min_i v_{i,j}, & g_j \in G^+ \\ \max_i v_{i,j}, & g_j \in G^- \end{cases}$$

Step 4: Calculate the Euclidian distances of each alternative for the ideal and anti-ideal solutions.

$$d_{ai}^* = \sqrt{\sum_{j=1}^{n} (v_{i,j} - v_j^*)^2},\ i = 1, 2, \cdots, m.$$

$$d_{ai}^- = \sqrt{\sum_{j=1}^{n} (v_{i,j} - v_j^-)^2},\ i = 1, 2, \cdots, m.$$

Step 5: Calculate the closeness coefficient of each alternative for the ideal solution based on the distances obtained in the previous Step.

$$Cl(a_i) = \frac{d_i^-}{d_i^* + d_i^-},\ i = 1, 2, \cdots, m$$

Step 6: Rank the alternatives in descending order of the closeness coefficient.

Sabokbar et al. (2016) proposed a variation of TOPSIS for sorting problems, called TOPSIS-Sort, detailed in Algorithm 2. Let $P = \{P_1, \cdots, P_q\}$ be a set of q profiles, and defining a profile $P_k = (\bar{P}_k, \underline{P}_k)$, where $\bar{P}_k$ is the upper limit of class C_k and $\underline{P}_k$ is the lower limit of class C_k. Alternatives, weights and criteria are defined in the same way as in Algorithm 1. Like the alternatives, the upper and lower limits of each profile receive performance values over te set of criteria. Algorithm 2 details the TOPSIS-Sort procedure.

Algorithm 2: TOPSIS-Sort Routine – Adapted from Sabokbar et al. (2016)

Step 1: Determine the Decision Matrix $X = [a_{i,j}]_{m\times n}$

Step 2: Establish a set of Profiles $P = \{(\bar{P}_1, \underline{P}_1), (\bar{P}_2, \underline{P}_2), \cdots, (\bar{P}_q, \underline{P}_q)\}$, where $\bar{P}_k$ and $\underline{P}_k$ are respectively the upper and lower limits of class C_k.

Step 3: Establish a new Decision Matrix $M = [M_{i,j}]_{(m+q-1)\times n}$ formed by the set of alternatives and profiles.

Step 4: Calculate the Normalized Decision Matrix $R = [r_{i,j}]_{(m+q-1)\times n}$ starting with the equations:

$$r_{i,j} = \frac{M_{i,j}}{\max_{1\le i\le (m+q-1)} M_{i,j}},\ \ i = 1, \cdots, (m+q-1);\ \ j = 1, \cdots, n.$$

Step 5: Conduct Steps 2 to 5 of traditional TOPSIS (Algorithm 1). Ideal and anti-ideal solutions are determined based on values from the weighted normalized matrix. Next, the Euclidean distances of each alternative, the upper limit profile, and the lower limit profile for the ideal and anti-ideal solutions are obtained. Finally, the closeness coefficients of each alternative $Cl(a_i)$, upper limit profile $Cl(\bar{P}_k)$ and lower limit profile $Cl(\underline{P}_k)$ are determined.

Step 6: Classify the alternatives by making comparisons between their closeness coefficients $Cl(a_i)$ and those of the upper $Cl(\bar{P}_k)$ and lower $Cl(\underline{P}_k)$ limits of the profiles.

$$a_i \in C_k\ \ iff\ \ Cl(\bar{P}_k) < Cl(a_i) < Cl(\underline{P}_k),\ \ i = 1, 2, \cdots, m;\ \ k = 1, 2 \cdots, q$$

When analyzing the TOPSIS-Sort procedure and the numerical example used to illustrate the method in (Sabokbar et al., 2016), some details stand out. In order to be able to sort all the alternatives among q ordered classes using Step 4, the lower limit of a profile should be equal to the upper level of the subsequent profile. Therefore, in Step 4, $Cl(P_k) = Cl(P_{k+1}^{-})$. Furthermore, there is no need to create an upper limit for the most preferred class and a lower limit for the least preferred class. On taking these observations into account, we can rewrite the procedure as follows: $p = (q - 1)$ profiles, where the profile P_k is both the lower limit of class C_k and the upper limit of class C_{k+1}. This explains why, in the numerical example (discussed in Section 2.1), 4 profiles are defined to represent 5 ordered classes. Also, that is the reason why only $(q - 1)$ instead of $2q$ rows are added to the initial set of alternatives in Step 3 of Algorithm 2. This change is one of the changes incorporated in TOPSIS-Sort-B. See Algorithm 3 in Section 3.

Another important aspect should be considered. As the profiles are defined as alternatives and the sorting process is conducted by comparing the closeness coefficients of alternatives and profiles, the procedure follows steps similar to those of the traditional ranking method, which results in the TOPSIS ranking reversal problem, and which has been analyzed in the literature and recently solved (de Aires, 2019; García-Cascales & Lamata, 2012). As to the sorting problematic, a ranking reversal involving an alternative and a profile may occur when the initial Decision Matrix is changed by adding or removing an alternative (and this change impacts the criteria domain). As a result, the class of the alternative involved in a ranking reversal with a profile would change. In the following section, a numerical example is used to analyze this problem.

2.2. Rank-reversal and sorting inconsistencies in TOPSIS

Table 1 presents the Decision Matrix used in the numerical example of Sabokbar et al. (2016), where TOPSIS-Sort was applied for the first time. In this application, there are 22 alternatives, 5 classes, and 4 profiles. Profile P_k works as a limit between classes C_k and C_{k+1}.

As the profiles are determined by the DM and each alternative is allocated by comparing its performance to the performance of the profiles, the inclusion of a new alternative should not change the class of any of the other m initial alternatives. On the other hand, the closeness coefficients of the profiles are calculated in a similar way to those of any other alternative, and a ranking reversal problem may occur when a new alternative expands the domain of a criterion (de Aires, 2019; García-Cascales & Lamata, 2012). Also, the domain of criteria can be compressed if an alternative is removed from the original set. Two domain variations are used to illustrate the problem:

- Variation #1: An external alternative is included: $a_{23} = \{70, 130, 49.2, 110, 190\}$, thereby expanding the domain of three criteria g_1, g_2 and g_4.
- Variation #2: Only half of the alternatives are considered, a_1 to a_{11}, thus compressing the domain of all the criteria.

Table 2 presents the original allocation and the changes with the two variations:

Essentially, as commented on above, these changes are caused when the domain of criteria varies. As a result, the values which are used to determine the change in ideal and anti-ideal solutions and new Euclidean distances and closeness coefficients are calculated, which may cause ranking reversals. Therefore, using the same decision matrix for a ranking problem of traditional TOPSIS could result in ranking reversals with domain changes, even between alternatives from different classes.

3. Novel sorting algorithms based on TOPSIS

In this Section, Algorithm 2 as set out by Sabokbar et al. (2016) is altered by using $p = q - 1$ boundary profiles for a problem of q ordered classes, including a solution for the ranking reversal problem (de Aires, 2019; García-Cascales & Lamata, 2012), and aspects of RTOPSIS (de Aires, 2019) for sorting problems are introduced, such as the interval normalization option. The procedure, called TOPSIS-Sort-B, is proposed in Algorithm 3. Also, we propose a new TOPSIS-Sort process, called TOPSIS-Sort-C, in which characteristic profiles are used for the sorting process instead of boundary profiles. In the case of TOPSIS-Sort-C, $p = q$ profiles are defined.

What prompted us to investigate TOPSIS-based sorting methods further was that they are applied widely and decision-makers find them

Table 1
Decision Matrix.

Alternatives	g_1(benefit)	g_2(benefit)	g_3(benefit)	g_4(benefit)	g_5(cost)
a_1	15.5	82.4	40.1	70.66	313.69
a_2	9.9	90.8	22	82.21	130.61
a_3	15.2	82.3	25.2	65.57	269.37
a_4	23.4	88.3	60.2	69.28	119.03
a_5	16.6	76.8	22.7	69.28	116.82
a_6	12.4	78.6	22.6	50.28	199.55
a_7	4	85	27.8	51.4	188.94
a_8	4.4	78.2	25.6	56.59	126.37
a_9	5.2	80.2	27.6	86.29	108.82
a_{10}	2.3	80.7	24.2	79.88	97.95
a_{11}	4.9	77.7	26.3	61.28	122.58
a_{12}	5.7	80.9	26.8	70.75	82.75
a_{13}	33.4	85.3	28.1	67.68	126.14
a_{14}	5.9	82.5	29.8	82.61	149.32
a_{15}	14	88.5	24.6	90.71	335.24
a_{16}	11.4	81.8	26.5	61.04	139.62
a_{17}	3.5	78.9	22.4	60.82	99.42
a_{18}	12.96	78.4	25.2	67.25	180.18
a_{19}	28	82.4	25.3	56.89	102.78
a_{20}	15.7	82.1	27.7	24.64	193.52
a_{21}	25	91.3	67.1	79.81	916.67
a_{22}	48	85.1	49.2	86.71	173.79
P_1	25	80	40	80	150
P_2	20	75	30	60	300
P_3	10	70	20	30	400
P_4	0	65	10	20	700
w_j	0.2	0.2	0.2	0.2	0.2

Source: Sabokbar et al. (2016).

Table 2
Classification of alternatives with changes in the domain of criteria.

Alternatives	Sorting Process		
	Original Set of Alternatives	Variation #1	Variation #2
a_1	C_2	C_2	C_2
a_2	C_2	C_2	C_3
a_3	C_3	C_3	C_3
a_4	C_1	C_1	C_1
a_5	C_2	C_2	C_2
a_6	C_3	C_3	C_3
a_7	C_3	C_3	C_3
a_8	C_3	C_3	C_3
a_9	C_2	C_2	C_3
a_{10}	C_3	C_2	C_3
a_{11}	C_3	C_3	C_3
a_{12}	C_3	C_2	–
a_{13}	C_2	C_2	–
a_{14}	C_2	C_2	–
a_{15}	C_2	C_2	–
a_{16}	C_3	C_2	–
a_{17}	C_3	C_3	–
a_{18}	C_3	C_2	–
a_{19}	C_2	C_2	–
a_{20}	C_3	C_3	–
a_{21}	C_2	C_2	–
a_{22}	C_1	C_1	–

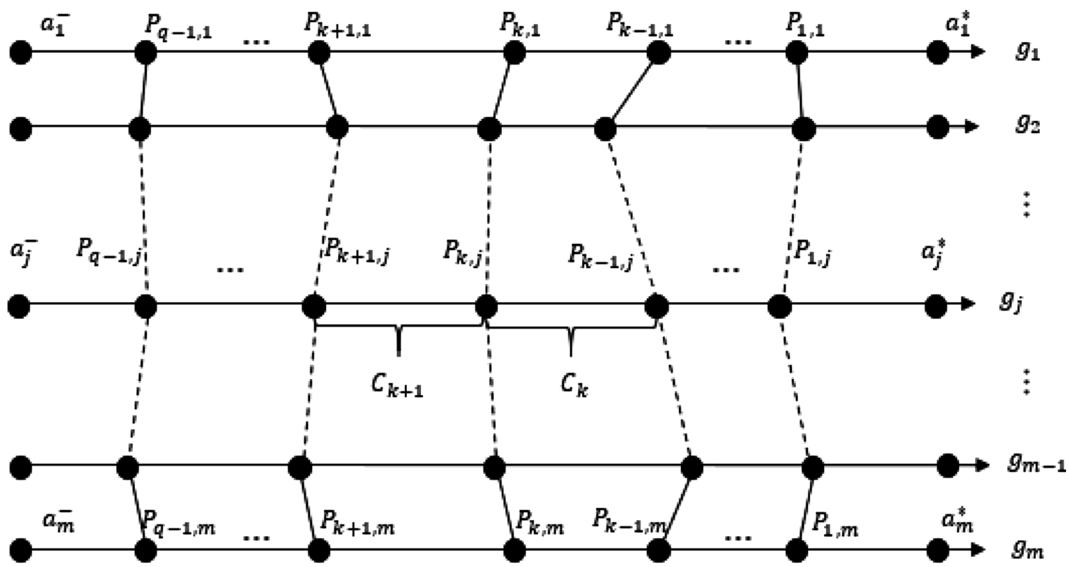

Fig. 1. Boundary Profiles.

easy to understand. TOPSIS-based methods include interesting aspects that can help solve MCDM/A problems, such as the fact that they consider the notion of measuring performance on alternatives based on their distance to ideal and non-ideal alternatives. This differentiates the method from other additive methods. Furthermore, TOPSIS-based sorting methods deal with a paradigm that is different from the one investigated in the classic PROMETHEE-based and ELECTRE-based sorting methods. While these latter two methods deal with outranking relations, TOPSIS-based methods consider the relative distance among the performances of the criteria. Asgharizadeh, Yazdi, and Balani (2019) classified 17 MCDM/A procedures using 7 criteria: "simplicity", "speed", "memory", "inputs", "logic", "quality", and "rate of growth". TOPSIS received the highest evaluation for "inputs" and "rate of growth" and was the third best evaluated method for the criterion of "simplicity". Moreover, in this paper, the methods we propose cover gaps found in the literature as described above.

Before describing the procedures, we highlight some assumptions and formalizations that are considered prior to applying the novel methods in sorting problems.

Assumption 1:. *The set of classes is pre-defined and the classes are ordered in terms of preference as follows:* $C_1 \succ C_2 \succ \cdots \succ C_q$.

Assumption 2:. *The domain of each criterion is known and can be represented by reference alternatives* a^* *and* a^-, *where* a_j^* *and* a_j^- *are respectively the largest and smallest possible values of criterion* g_j.

Assumption 3.1 (*(valid only for TOPSIS-Sort-B):*). *The limit between two consecutive classes* C_k *and* C_{k+1} *is defined by one boundary profile* P_k, *where the profile* P_k *is both the lower limit of class* C_k *and the upper limit of class* C_{k+1}.*Fig. 1 illustrates an example of how boundary profiles are constructed. The value of* $P_{k,j}$ *should represent what is expected to be the limit value between two consecutive categories* (C_k *and* C_{k+1}) *for criterion* g_j.

Assumption 3.2 (*(valid only for TOPSIS-Sort-C):*). *Each class* C_k *is defined by one characteristic profile* P_k, *which is the most representative reference alternative of the class.Fig. 2 shows a graphical representation of the concept of characteristic profiles. As illustrated, now the value of* $P_{k,j}$ *should represent what is expected to be a typical evaluation of an alternative from class* C_k *for criterion* j.

Assumption 4:. *For any pair of consecutive profiles, the profile* P_k *dominates the profile* P_{k+1}. *In other words,* $P_{k,j} \succsim P_{k+1,j}$, *where the strict preference* ($\succ$) *holds for at least one criterion.*

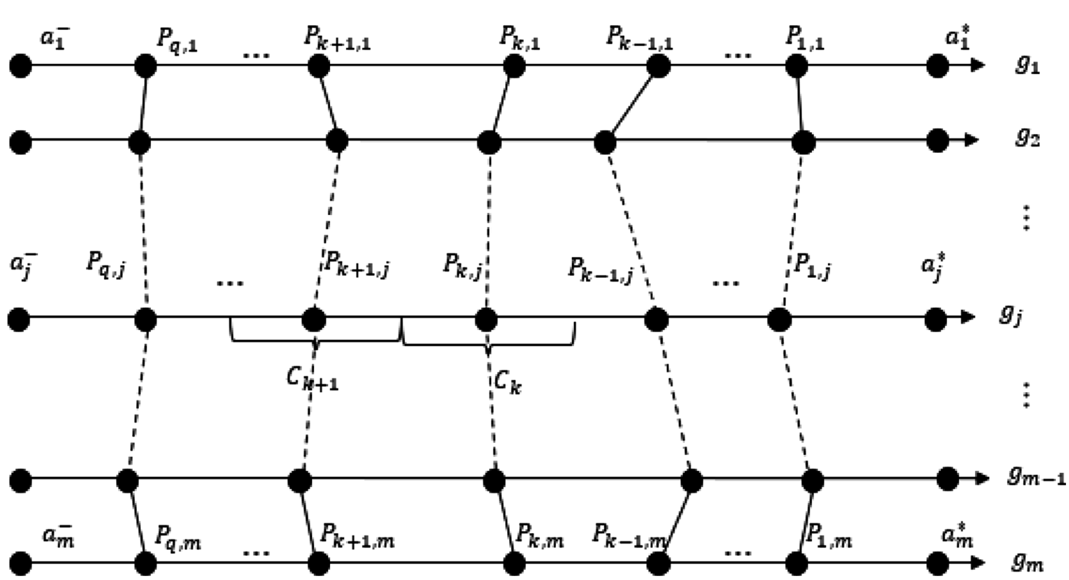

Fig. 2. Characteristic Profiles.

3.1. TOPSIS-Sort-B

Let $P = \{P_1, \cdots, P_p\}$ be a set of $p = q - 1$ profiles, in which profile P_k is the boundary profile that defines the limit between classes C_k and C_{k+1}. Algorithm 3 details the TOPSIS-Sort-B method.

Algorithm 3: TOPSIS-Sort-B Routine

Step 1: Determine the Decision Matrix $X = [a_{i,j}]_{m\times n}$

Step 2: Establish the Boundary Profiles Matrix $P = [P_{k,j}]_{p\times n}$, where $p = q - 1$

Step 3: Determine the domain of each criterion, thereby creating maximum and minimum values that could be reached by an alternative in each criterion. Later, these values will play the role of Ideal and Nonideal solutions. The domain is represented by the matrix $D = \begin{bmatrix} a_1^* & \cdots & a_n^* \\ a_1^- & \cdots & a_n^- \end{bmatrix}$, where a_j^* and a_j^- are respectively the largest and smallest possible value of criterion g_j.

Step 4: Establish the Complete Decision Matrix $M = [M_{i,j}]_{(m+p+2)\times n} = \begin{bmatrix} X \\ P \\ D \end{bmatrix}$, by vertically concatenating $X = [a_{i,j}]_{m\times n}$, $P = [P_{k,j}]_{p\times n}$, and $D = \begin{bmatrix} a_1^* & \cdots & a_n^* \\ a_1^- & \cdots & a_n^- \end{bmatrix}$.

Step 5: Determine the weighted and normalized Decision Matrix $V = [v_{i,j}]_{(m+p+2)\times n}$

Step 5.1: Normalize the Decision Matrix M starting with the equations:

Option 1 (Normalization by the Max):

$$r_{i,j} = \frac{M_{i,j}}{a_j^*},\ i = 1, \cdots, (m + p + 2);\ j = 1, \cdots, n$$

Option 2 (Interval Normalization):

$$r_{i,j} = \frac{M_{i,j} - a_j^-}{a_j^* - a_j^-},\ i = 1, \cdots, (m + p + 2);\ j = 1, \cdots, n$$

Step 5.2: Calculate the weighted and normalized decision matrix $V = [v_{ij}]_{(m+p+2)\times n}$

$$v_{i,j} = w_j r_{i,j},\ i = 1, 2, \ldots, (m + p + 2);\ j = 1, 2, \ldots, n$$

Where: $\sum_{j=1}^{n} w_j = 1$

Step 6: Determine the ideal and anti-ideal solutions.

$$v^* = [v_1^*, v_2^*, \cdots, v_n^*],\ v_j^* = \begin{cases} \max\limits_i v_{i,j},\ g_j \in G^+ \\ \min\limits_i v_{i,j},\ g_j \in G^- \end{cases}$$

$$v^- = [v_1^-, v_2^-, \cdots, v_n^-],\ v_j^- = \begin{cases} \min\limits_i v_{i,j},\ g_j \in G^+ \\ \max\limits_i v_{i,j},\ g_j \in G^- \end{cases}$$

Step 7: Calculate the Euclidian distances of each alternative and profile for the ideal and anti-ideal solutions.

$$d_{a_i}^* = \sqrt{\sum_{j=1}^{n} (v_{i,j} - v_j^*)^2},\ i = 1, 2, \cdots, m$$

$$d_{a_i}^- = \sqrt{\sum_{j=1}^{n} (v_{i,j} - v_j^-)^2},\ i = 1, 2, \cdots, m$$

$$d_{P_k}^* = \sqrt{\sum_{j=1}^{n} (v_{i,j} - v_j^*)^2},\ k = 1, 2, \cdots, p;\ i = k + m$$

$$d_{P_k}^- = \sqrt{\sum_{j=1}^{n} (v_{i,j} - v_j^-)^2},\ k = 1, 2, \cdots, p;\ i = k + m$$

Step 8: Calculate the closeness coefficient of each alternative and profile for the ideal solution based on the distances obtained in the previous Step.

$$Cl(a_i) = \frac{d_{a_i}^-}{d_{a_i}^* + d_{a_i}^-},\ i = 1, 2, \cdots, m$$

$$Cl(P_k) = \frac{d_{P_k}^-}{d_{P_k}^* + d_{P_k}^-},\ k = 1, 2, \cdots p$$

Step 9: Classify the alternatives by making comparisons between their closeness coefficients $Cl(a_i)$, and those of the profiles $Cl(P_k)$.

$a_i \in C_1$ *iff* $Cl(a_i) \geq Cl(P_1)$

$a_i \in C_k$ *iff* $Cl(P_{k-1}) > Cl(a_i) \geq Cl(P_k),\ k = 2, \cdots, (q - 1)$

$a_i \in C_q$ *iff* $Cl(a_i) < Cl(P_{q-1})$

TOPSIS-Sort-B differs from the original TOPSIS-Sort method set out in Algorithm 2 because it:

i. Defines only $q - 1$ profiles;
ii. Adds Step 3 to prevent sorting changes derived from the TOPSIS ranking reversal problem. This step is used in RTOPSIS, which was proposed by (Aires & Ferreira, 2019) as the solution for the ranking reversal in TOPSIS;
iii. Adds the option of an interval normalization in Step 5.1;

For the application of TOPSIS-Sort-B, the analyst and the DM take a constructive approach to determining the boundary profiles. The domain for each criterion should be defined by an expert. The definition of only $q - 1$ boundary profiles makes it easier to understand the role that they play, which is to specify the limits between two consecutive categories. Therefore, the DM does not need to define an upper limit profile for the best category and a lower limit profile for the worst one. In contrast, the limits of each criterion are specified as a result of having defined the respective domain in Step 3. Furthermore, the inclusion of Step 3 represents an important improvement to the TOPSIS-Sort algorithm, with regard to the change in classification problem discussed in Section 2.1. The fictitious alternatives ($a^*$and a^-), defined in this step will set the range of acceptable values for the application, and will function as ideal and anti-ideal solutions. Therefore, as neither the inclusion nor the removal of alternatives impacts the domain, the ideal and anti-ideal solutions do not change, and ranking reversals are avoided.

3.2. TOPSIS-Sort-C

It is often easier to define a class by using a characteristic profile than by defining boundary profiles in order to understand the limits between consecutive classes. For instance, when a financial analyst aims to define if a sovereign bond should be allocated to a risky or to a safe category, he/she could compare the characteristics of the country by using a set of financial criteria applied to a low-risk bond example or a high-risk country. On the other hand, under a boundary profile perspective, the financial analyst would think about a profile that is at the limit between the two defined classes, which could be less intuitive. Taking this into consideration, methods to sort alternatives using characteristic profiles have been proposed in the MCDM/A literature, such as the ELECTRE-TRI-C and ELECTRE-TRI-nC variations of the traditional ELECTRE-TRI sorting method (Almeida-Dias et al., 2010, 2012).

In this paper, we propose TOPSIS-Sort-C in order to address this type of sorting problem from a TOPSIS perspective. This method should be applied when a characteristic profile/action is defined so as to explain each pre-defined class. Like the previous method, TOPSIS-Sort-B, TOPSIS-Sort-C should be applied for preference ordered classification problems.

Let $P = \{P_1, \cdots, P_p\}$ be a set of $p = q$ profiles, in which profile P_k is a characteristic profile of class C_k. Algorithm 4 details the TOPSIS-Sort-C method.

Algorithm 4: TOPSIS-Sort-C Routine

Step 1: Determine the Decision Matrix $X = [a_{i,j}]_{m\times n}$

Step 2: Establish the Boundary Profiles Matrix $P = [P_{k,j}]_{p\times n}$, where $p = q$

Step 3: Determine the domain of each criterion, creating maximum and minimum values that could be reached by an alternative in each criterion. Later, these values will play the role of Ideal and Nonideal solutions. The domain is represented by the matrix $D = \begin{bmatrix} a_1^* & \cdots & a_n^* \\ a_1^- & \cdots & a_n^- \end{bmatrix}$, where a_j^* and a_j^- are respectively the largest and smallest possible value of criterion g_j.

Step 4: Establish the Complete Decision Matrix $M = [M_{i,j}]_{(m+p+2)\times n} = \begin{bmatrix} X \\ P \\ D \end{bmatrix}$, by vertically concatenating $X = [a_{i,j}]_{m\times n}$, $P = [P_{k,j}]_{p\times n}$, and $D = \begin{bmatrix} a_1^* & \cdots & a_n^* \\ a_1^- & \cdots & a_n^- \end{bmatrix}$.

Step 5: Determine the weighted and normalized Decision Matrix $V = [v_{i,j}]_{(m+p+2)\times n}$

Step 5.1: Normalize the Decision Matrix M starting with the equations:

Option 1 (Normalization by the Max):

$r_{i,j} = \frac{M_{i,j}}{a_j^*}$, $i = 1, \cdots, (m + p + 2); j = 1, \cdots, n$

Option 2 (Interval Normalization):

$r_{i,j} = \frac{M_{i,j} - a_j^-}{a_j^* - a_j^-}$, $i = 1, \cdots, (m + p + 2); j = 1, \cdots, n$

Step 5.2: Calculate the weighted and normalized decision matrix $V = [v_{ij}]_{(m+p+2)\times n}$

$v_{i,j} = w_j r_{i,j}$, $i = 1, 2, ..., (m + p + 2); j = 1, 2, ..., n$

Where: $\sum_{j=1}^{n} w_j = 1$

Step 6: Determine the ideal and anti-ideal solutions.

$$v^* = [v_1^*, v_2^*, \cdots, v_n^*],\ v_j^* = \begin{cases} \max\limits_i v_{i,j},\ g_j \in G^+ \\ \min\limits_i v_{i,j},\ g_j \in G^- \end{cases}$$

$$v^- = [v_1^-, v_2^-, \cdots, v_n^-],\ v_j^- = \begin{cases} \min\limits_i v_{i,j},\ g_j \in G^+ \\ \max\limits_i v_{i,j},\ g_j \in G^- \end{cases}$$

Step 7: Calculate the Euclidian distances of each alternative and profile for the ideal and anti-ideal solutions.

$d_{a_i}^* = \sqrt{\sum_{j=1}^{n}(v_{i,j} - v_j^*)^2}$, $i = 1, 2, \cdots, m$

$d_{a_i}^- = \sqrt{\sum_{j=1}^{n}(v_{i,j} - v_j^-)^2}$, $i = 1, 2, \cdots, m$

$d_{P_k}^* = \sqrt{\sum_{j=1}^{n}(v_{i,j} - v_j^*)^2}$, $k = 1, 2, \cdots, p; i = k + m$

$d_{P_k}^- = \sqrt{\sum_{j=1}^{n}(v_{i,j} - v_j^-)^2}$, $k = 1, 2, \cdots, p; i = k + m$

Step 8: Calculate the closeness coefficient of each alternative and profile for the ideal solution based on the distances obtained in the previous Step.

$Cl(a_i) = \frac{d_{a_i}^-}{d_{a_i}^* + d_{a_i}^-}$, $i = 1, 2, \cdots, m$

$Cl(P_k) = \frac{d_{P_k}^-}{d_{P_k}^* + d_{P_k}^-}$, $k = 1, 2, \cdots p$

Step 9: Classify the alternatives by making comparisons between their closeness coefficients $Cl(a_i)$, and those of the profiles $Cl(P_k)$.

$a_i \in C_1$ iff $|Cl(a_i) - Cl(P_1)| \le |Cl(a_i) - Cl(P_2)|$, $i = 1, \cdots, m$

$a_i \in C_k$ iff $\begin{cases} |Cl(a_i) - Cl(P_k)| < |Cl(a_i) - Cl(P_{k-1})| \\ |Cl(a_i) - Cl(P_k)| \le |Cl(a_i) - Cl(P_{k+1})| \end{cases}$, $i = 1, \cdots, m;\ k = 2, \cdots, (q - 1)$

$a_i \in C_q$ iff $|Cl(a_i) - Cl(P_q)| < |Cl(a_i) - Cl(P_{q-1})|$, $i = 1, \cdots, m$

The DM and the analyst establish the characteristic profiles in Step 2 and they take a constructive approach to this task. On analyzing Step 4 of Algorithm 4, notice that the classification rule changes in comparison with Algorithm 3. Now, the allocation is based on the differences between the closeness coefficients of each alternative and those of the profiles. Alternatives are allocated to the class for which the closeness coefficient of the respective profile has the most similar value.

3.3. Properties of TOPSIS-Sort-B and TOPSIS-Sort-C

In order to guarantee that the proposed methods are stable and make sense considering the features of sorting problems and what is expected in these situations, it is important to investigate whether or not the methods respect some properties. Therefore, we show properties of TOPSIS-Sort-B and TOPSIS-Sort-C. The desired properties for sorting procedures are properties 1, 2, 3, and 6 as set out by (Almeida-Dias, Figueira, & Roy, 2010) when introducing the ELECTRE-TRI-C method. In addition, we show that Property 4 protects TOPSIS-Sort-B and TOPSIS-Sort-C from ranking reversals and we establish that Property 5 is a stronger version of the stability property that is valid for TOPSIS-Sort-B. Appendix A details the verification of each property.

Property 1. - ***Conformity***: *any alternative a_i similar to the boundary/characteristic profile P_k ($a_{i,j} = P_{k,j}, \forall j$) must be allocated to class C_k.*

Property 2. - ***Homogeneity***: *two different alternatives must be allocated to the same class if they are equally distant from the ideal (v^*) and anti-ideal (v^-) alternatives.*

Property 3. - ***Monotonicity***: *if alternative a_i dominates alternative a_s, then it must be allocated to a class at least as good as the class to which a_s is allocated.*

Property 4. – ***Irreversibility*** **(*ranking reversal protection*)**: *The classification of an alternative a_i must not be affected by the inclusion of one or more new alternatives in the initial set A or by the elimination one or more alternatives that were initially considered in set A.*

Definition 1. - ***Merging Operation***: *Two consecutive classes, C_k and C_{k+1}, will be merged to become a new class $C_k^{'}$* (Almeida-Dias et al., 2010).

- *In the case of TOPSIS-Sort-B, the boundary profile P_k is excluded since there is no longer a frontier between C_k and C_{k+1}. P_{k+1} is renamed as $P_k^{'}$.*
- *In the case of TOPSIS-Sort-C, a new characteristic profile $P_k^{'}$ substitutes the two profiles P_k and P_{k+1}, which were defined so as to respect Assumption 1 and Assumption 4 (the relation must be strict for at least one criterion to guarantee dominance):*

$$\begin{cases} P_{k,j} \ge P_{k,j}^{'} \ge P_{k+1,j},\ \forall\ j \in G^+ \\ P_{k,j} \le P_{k,j}^{'} \le P_{k+1,j},\ \forall\ j \in G^- \end{cases}$$

Definition 2. – ***Splitting Operation***: *A class C_k will be split into two new consecutive classes, $C_k^{'}$ and $C_k^{''}$* (Almeida-Dias et al., 2010).

- *In the case of TOPSIS-Sort-B, a new boundary profile $P_k^{'}$ is needed in order to define the limit between $C_k^{'}$ and $C_k^{''}$, thereby respecting Assumption 1 and Assumption 4:*

$$\begin{cases} P_{k-1,j} \ge P_{k,j}^{'} \ge P_{k,j},\ \forall\ j \in G^+ \\ P_{k-1,j} \le P_{k,j}^{'} \le P_{k,j},\ \forall\ j \in G^- \end{cases}$$

- *In the case of TOPSIS-Sort-C, two new profiles ($P_k^{'}$ and $P_k^{''}$) are defined to replace P_k, thereby respecting Assumption 1 and Assumption 4:*

$$\begin{cases} P_{k-1,j} \ge P_{k,j}^{'} \ge P_{k,j}^{''} \ge P_{k+1,j},\ \forall\ j \in G^+ \\ P_{k-1,j} \le P_{k,j}^{'} \le P_{k,j}^{''} \le P_{k+1,j},\ \forall\ j \in G^- \end{cases}$$

Property 5. – ***Strong Stability*** **(*applied for TOPSIS-Sort-B*)**: *After applying a merging or a splitting operation, alternatives previously allocated to non-modified classes will be allocated to the same class. Alternatives previously allocated to the merged/split class will be allocated to a new class that will be constructed after the operation.*

Property 6. – ***Stability***: (Almeida-Dias et al., 2010) *After applying a merging or a splitting operation, alternatives which were previously allocated to a class that was non-adjacent to the modified one will be allocated to the same class. Alternatives which were previously allocated to a class that was adjacent to the one modified will be allocated to the same class or to a new one after the operation. Alternatives previously allocated to the merged/split class will be allocated to the new class or to an adjacent one.*

4. Numerical application for evaluating economic freedom

Evaluating country risk by using ranking and sorting methods is a

Table 3
Set of criteria – economic freedom application.

	Criteria		Detailed description
Rule of law	g_1	Property Rights	Secure property rights give citizens the confidence to undertake entrepreneurial activity, save their income, and make long-term plans because they know that their income, savings, and property (both real and intellectual) are safe from unfair expropriation or theft.
	g_2	Judicial Effectiveness	Judicial effectiveness requires efficient and fair judicial systems to ensure that laws are fully respected, with appropriate legal actions taken against violations.
	g_3	Government Integrity	Practices that allow some individuals or special interests to gain government benefits at the expense of others are grossly incompatible with the principles of fair and equal treatment that are essential ingredients of an economically free society.
Size of Government	g_4	Tax Burden	Governments that permit individuals and businesses to keep and manage a larger share of their income and wealth for investment and reward purposes and thus to maximize opportunities created by greater economic freedom.
	g_5	Government Spending	Excessive government spending runs a great risk of crowding out private economic activity.
	g_6	Fiscal Health	High levels of public debt may have numerous negative impacts such as raising interest rates, crowding out private investment, and limiting government's flexibility in responding to economic crises.
Regulatory Efficiency	g_7	Business Freedom	Burdensome and redundant regulations are the most common barriers to the free conduct of entrepreneurial activity. By increasing the costs of production, regulations can make it difficult for entrepreneurs to succeed in the marketplace.
	g_8	Labor Freedom	The core principle of any economically free market is voluntary exchange. That is just as true in the labor market as it is in the market for goods.
	g_9	Monetary Freedom	Whether acting as entrepreneurs or as consumers, economically free people need a steady and reliable currency as a medium of exchange, unit of account, and store of value. Without monetary freedom, it is difficult to create long-term value or amass capital.
Open Markets	g_{10}	Trade Freedom	Many governments place restrictions on their citizens' ability to interact freely as buyers or sellers in the international marketplace.
	g_{11}	Investment Freedom	A free and open investment environment provides maximum entrepreneurial opportunities and incentives for expanded economic activity, greater productivity, and job creation.
	g_{12}	Financial Freedom	An accessible and efficiently functioning formal financial system ensures the availability of diversified savings, credit, payment, and investment services to individuals and businesses.

Source: The Heritage Foundation (2019)

classic application of MCDM/A in finance (Kosmidou, Doumpos, & Zopounidis, 2008; Zopounidis, Galariotis, Doumpos, Sarri, & Andriosopoulos, 2015). Commonly, financial indexes are used as criteria to evaluate the financial health of countries, and this information is important as investors use it to analyze returns on sovereign bonds and investment risk (Becerra-Fernandez, Zanakis, & Walczak, 2002; de Lima Silva, Silva, Silva, Ferreira, & de Almeida-Filho, 2018; Doumpos, Pentaraki, Zopounidis, & Agorastos, 2001; Greco, Matarazzo, Slowinski, & Zanakis, 2011; Zopounidis & Doumpos, 2002b). In this paper, we analyze countries from a different perspective, using the proposed TOPSIS-Sort-B and TOPSIS-Sort-C to evaluate their economic freedom. Socioeconomic data are a very interesting to deploy MCDM approaches, similarly as in many cases found on the literature (do Carvalhal Monteiro et al., 2020, 2018, 2019; Mazziotta & Pareto, 2015; Tervonen, Kingdom, Dias, & Lahdelma, 2007).

Annually, The Heritage Foundation publishes its Index of Economic Freedom (Foundation, 2019). 2019 saw the 25th anniversary of this index, which evaluates various countries over a set of 12 criteria, grouped into 4 categories: rule of law; size of government; regulatory efficiency; and open markets. Table 3 presents the set of criteria in addition to giving a detail description of their meaning for the decision context presented by (Foundation, 2019).

For each criterion, the Heritage Foundation has a set of subfactors, represented by quantitative and qualitative indexes, which are used to obtain a criterion score that ranges between 0 and 100 for each country. Each criterion has specific rules for determining this score, details of which are available in (Foundation, 2019). After obtaining the performance of the alternatives for the 12 criteria, a simple average is calculated, and the result for each country represents its economic freedom score. Thus, the index considers the criteria as being equally

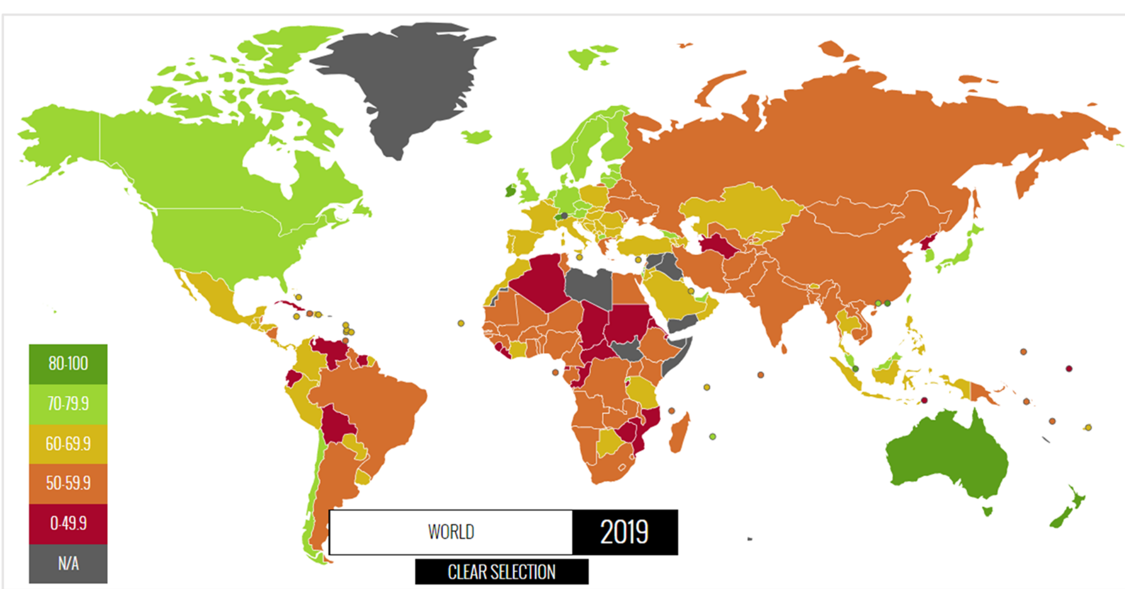

Fig. 3. Heat Map – Economic Freedom. Source: The Heritage Foundation (2019).

Table 4
Categories of Economic Freedom.

Category		Boundary Profiles
C_1	Free	$80 \leq score \leq 100$
C_2	Mostly Free	$70 \leq score < 80$
C_3	Moderately Free	$60 \leq score < 70$
C_4	Mostly Unfree	$50 \leq score < 60$
C_5	Repressed	$0 \leq score < 50$

weighted and the overall score also ranges between 0 and 100.

In this numerical application, our objective is to illustrate the functionality of the proposed TOPSIS-based methods when a structured decision process is used. We adopted the Heritage Foundation decision process over the choice of criteria and weights to simulate this problem of sorting countries according to their economic freedom. Furthermore, we consider it makes sense to use the same criteria and data as The Heritage Foundation does since this makes it more straightforward both to conduct posterior analysis and to compare the assignments of different procedures with the original assignments. Details beyond those given in Table 3 on the meaning of the criteria are available in the Foundation's annual report (Foundation, 2019).

In 2019, the Heritage Foundation evaluated 180 countries and ranked them using the calculated economic freedom index. In addition, the Foundation assigned the countries to five categories of economic freedom according to their overall score. Fig. 3 shows the results of assigning the alternatives on a heat map.

In Fig. 3, the result of the sorting process are illustrated on a map of the world which is supported by the key on the left. The colors represent the five different categories while gray shows countries for which there are no data while the values shown in the key act as boundary profiles. Table 4 details the categories considered and the boundaries used by the Heritage Foundation to allocate the alternatives.

The above decision problem can be described as an MCDM/A sorting problem. Now, we apply TOPISIS-Sort-B and TOPSIS-Sort-C to assign the same set of 180 countries to one of the five categories described in Table 4. As countries with the highest scores are considered to be freer economically, we can define the ordered classes: $C_1 \succ C_2 \succ C_3 \succ C_4 \succ C_5$. Since the performance of the countries in each criterion ranges between 0 and 100, these values were used to represent the domain of criteria. To make comparisons between the results obtained with the proposed methods and the original assignments of the Heritage Foundation, we set parameters for TOPSIS-Sort-B and TOPSIS-Sort-C that are similar to those used by the Foundation. First, we regarded criteria are being equally weighted. For TOPSIS-Sort-B we defined 4 boundary profiles by respecting the intervals used by the Foundation and which are shown in Table 4. This represents Scenario #1. In the case of TOPSIS-Sort-C, two other scenarios were considered. In Scenario #2, the 5 characteristic profiles were defined as the median of the interval associated to each category in Table 4. In Scenario #3, the characteristic profiles are the average evaluation of the alternatives originally allocated to the different categories by the Heritage Foundation.

Table 5 details the average performances of the countries for each criterion per category in line with how the Heritage Foundation had assigned them. In this table, criteria g_4 (tax burden) and g_5(government spending) are highlighted because their values are not in descending order. On the other hand, these criteria have positive monotonicity of preferences (higher scores are preferable). As there are 12 equally weighted criteria, and the classification is made by simple average, these situations occurred because the performances of the alternatives in the criteria allowed compensations. Also, we can see that the ranges of the 5 averages for g_4 and g_5 are narrower than those for the other criteria. If these values were used to determine the characteristic profiles in TOPSIS-Sort-C, inconsistencies would occur. For instance, a preferable profile would receive a smaller value for a given criterion: $P_{2,4} < P_{3,4}$. To overcome this situation, we decided to order the values of these two criteria when defining the characteristic profiles as presented in detail in Table 6. Therefore, for all criteria and profiles considered in this application: $P_{k,j} > P_{k+1,j}$.

It is important to mention that in this illustrative application we use the Heritage Foundation's original assessment to mimic a DM, so the above adjustments were important to prevent inconsistencies. In contrast, for an application with a real DM, the characteristic profiles will be chosen in a constructive way, in which the analyst will make sure that profiles from preferable classes receive better scores than profiles from less preferable classes. Still with regard to criteria g_4 and g_5 in Table 5, MCDM/A methods such as rule-based and preference disaggregation approaches could detect this behavior by means of a pre-analysis of the set of reference alternatives. As we discuss in the conclusions, statistical tools and these types of methods should be applied in future studies to analyze if this behavior holds for prior years.

Table 6 presents the Decision Matrix with a subset of 10 alternatives (180 countries are too large a number for this paper to analyze), the domain chosen and the profiles used in this application. The set of criteria is the same as that presented in Table 3. The complete Decision Matrix is available as supplementary material.

Table 6 details how the characteristics profiles changed between Scenarios #2 and #3. As explained above, while Scenario #2 considers only the median of each interval from Table 4, in Scenario #3 the characteristic profiles considered the real performances of the countries assigned to the categories in each criterion. It is expected that these differences impact the final classification of alternatives as the medians of the intervals may not necessarily represent the reality of the alternatives. For example, the average values for only two criteria (g_6 and g_{10}) were considered greater than the median of the intervals for category C_1.

After defining the Decision Matrix, the data should be normalized and weighted as described in Step 5 of Algorithm 3. In this application, the domain is the same for all criteria. Therefore, the two options of normalization procedures give the same results. Table 7 presents the normalized decision matrix, again considering only a subset of 10 alternatives. The complete table is available as supplementary material.

Table 8 details the results obtained by using the methods proposed for the three scenarios studied and the original assignments made by the Heritage Foundation for a subset of 10 alternatives. Furthermore, the Euclidean distances calculated for the ideal and anti-ideal solutions and the closeness coefficients of each alternative and profile are also shown in Table 8. The complete table with the results for the 180 alternatives is available as supplementary material.

Table 5
Average performance of countries using the considering assignments made by the Heritage Foundation.

Category	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
C_1	89.32	81.35	86.92	77.35	72.28	91.68	87.50	83.13	86.33	90.53	85.00	83.33
C_2	78.98	69.33	69.46	72.82	55.38	88.28	80.11	66.72	80.81	85.01	77.76	68.97
C_3	57.45	46.40	42.19	79.74	63.57	75.91	68.36	61.18	78.70	78.73	68.14	55.76
C_4	41.70	38.11	31.46	78.98	71.52	57.77	58.51	58.02	72.60	69.26	45.31	37.34
C_5	30.07	23.68	24.95	71.05	56.61	34.46	41.05	44.03	64.46	59.60	32.27	25.91

Table 6
Decision Matrix with a Subset of Alternatives.

Alternative	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
Afghanistan	19.6	29.6	25.2	91.7	80.3	99.3	49.2	60.4	76.7	66.0	10	10
Albania	54.8	30.6	40.4	86.3	73.9	80.6	69.3	52.7	81.5	87.8	70	70
Algeria	31.6	36.2	28.9	76.4	48.7	18.7	61.6	49.9	74.9	67.4	30	30
Angola	35.9	26.6	20.5	83.9	80.7	58.2	55.7	58.8	55.4	61.2	30	40
Argentina	47.8	44.5	33.5	69.3	49.5	33.0	56.4	46.9	60.2	70.0	55	60
Armenia	57.2	46.3	38.6	84.7	79.0	53.0	78.3	71.4	77.8	80.8	75	70
Australia	79.1	86.5	79.9	62.8	60.1	86.2	88.3	84.1	86.6	87.6	80	90
Austria	84.2	71.3	77.4	50.5	24.5	85.5	74.9	68.7	81.5	86.0	90	70
Azerbaijan	59.1	53.1	44.7	87.5	59.5	89.4	69.5	63.9	63.0	74.6	60	60
Bahamas	42.2	46.9	43.7	97.3	86.8	65.7	68.5	67.5	78.1	47.8	50	60
Domain of the Criteria												
a^*	100	100	100	100	100	100	100	100	100	100	100	100
a^-	0	0	0	0	0	0	0	0	0	0	0	0
Scenario #1: TOPSIS-Sort-B Boundary Profiles												
P_1	80	80	80	80	80	80	80	80	80	80	80	80
P_2	70	70	70	70	70	70	70	70	70	70	70	70
P_3	60	60	60	60	60	60	60	60	60	60	60	60
P_4	50	50	50	50	50	50	50	50	50	50	50	50
Scenario #2: TOPSIS-Sort-C Characteristic Profiles (considering the median of the intervals)												
P_1	90	90	90	90	90	90	90	90	90	90	90	90
P_2	75	75	75	75	75	75	75	75	75	75	75	75
P_3	65	65	65	65	65	65	65	65	65	65	65	65
P_4	55	55	55	55	55	55	55	55	55	55	55	55
P_5	25	25	25	25	25	25	25	25	25	25	25	25
Scenario #3: TOPSIS-Sort-C Characteristic Profiles (considering the average scores of the original allocations)												
P_1	89.32	81.35	86.92	79.74	72.28	91.68	87.50	83.13	86.33	90.53	85.00	83.33
P_2	78.98	69.33	69.46	78.98	71.52	88.28	80.11	66.72	80.81	85.01	77.76	68.97
P_3	57.45	46.40	42.19	77.35	63.57	75.91	68.36	61.18	78.70	78.73	68.14	55.76
P_4	41.70	38.11	31.46	72.82	56.61	57.77	58.51	58.02	72.60	69.26	45.31	37.34
P_5	30.07	23.68	24.95	71.05	55.38	34.46	41.05	44.03	64.46	59.60	32.27	25.91

Table 7
Weighted and Normalized Decision Matrix.

Alternative	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
Afghanistan	0.016	0.025	0.021	0.076	0.067	0.083	0.041	0.050	0.064	0.055	0.008	0.008
Albania	0.046	0.026	0.034	0.072	0.062	0.067	0.058	0.044	0.068	0.073	0.058	0.058
Algeria	0.026	0.030	0.024	0.064	0.041	0.016	0.051	0.042	0.062	0.056	0.025	0.025
Angola	0.030	0.022	0.017	0.070	0.067	0.049	0.046	0.049	0.046	0.051	0.025	0.033
Argentina	0.040	0.037	0.028	0.058	0.041	0.028	0.047	0.039	0.050	0.058	0.046	0.050
Armenia	0.048	0.039	0.032	0.071	0.066	0.044	0.065	0.060	0.065	0.067	0.063	0.058
Australia	0.066	0.072	0.067	0.052	0.050	0.072	0.074	0.070	0.072	0.073	0.067	0.075
Austria	0.070	0.059	0.065	0.042	0.020	0.071	0.062	0.057	0.068	0.072	0.075	0.058
Azerbaijan	0.049	0.044	0.037	0.073	0.050	0.075	0.058	0.053	0.053	0.062	0.050	0.050
Bahamas	0.035	0.039	0.036	0.081	0.072	0.055	0.057	0.056	0.065	0.040	0.042	0.050
Domain of the Criteria												
a^*	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083
a^-	0	0	0	0	0	0	0	0	0	0	0	0
Scenario #1: TOPSIS-Sort-B Boundary Profiles												
P_1	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067
P_2	0.058	0.058	0.058	0.058	0.058	0.058	0.058	0.058	0.058	0.058	0.058	0.058
P_3	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
P_4	0.042	0.042	0.042	0.042	0.042	0.042	0.042	0.042	0.042	0.042	0.042	0.042
Scenario #2: TOPSIS-Sort-C Characteristic Profiles (considering the median of the intervals)												
P_1	0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.075
P_2	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063
P_3	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.054
P_4	0.046	0.046	0.046	0.046	0.046	0.046	0.046	0.046	0.046	0.046	0.046	0.046
P_5	0.021	0.021	0.021	0.021	0.021	0.021	0.021	0.021	0.021	0.021	0.021	0.021
Scenario #3: TOPSIS-Sort-C Characteristic Profiles (considering the average scores of the original allocations)												
P_1	0.074	0.068	0.072	0.066	0.060	0.076	0.073	0.069	0.072	0.075	0.071	0.069
P_2	0.066	0.058	0.058	0.066	0.060	0.074	0.067	0.056	0.067	0.071	0.065	0.057
P_3	0.048	0.039	0.035	0.064	0.053	0.063	0.057	0.051	0.066	0.066	0.057	0.046
P_4	0.035	0.032	0.026	0.061	0.047	0.048	0.049	0.048	0.061	0.058	0.038	0.031
P_5	0.025	0.020	0.021	0.059	0.046	0.029	0.034	0.037	0.054	0.050	0.027	0.022

Table 8
Sorting Results.

Alternative	d_i^*	d_i^-	Cl_i	TOPSIS-Sort-B	TOPSIS-Sort-C		Original Assignments
					Sc#2	Sc#3	
Afghanistan	0.1657	0.1730	0.5109	4	4	4	4
Albania	0.1091	0.1984	0.6453	3	3	3	3
Algeria	0.1647	0.1442	0.4667	5	4	5	5
Angola	0.1533	0.1564	0.5050	4	4	4	4
Argentina	0.1420	0.1542	0.5207	4	4	4	4
Armenia	0.1022	0.1998	0.6615	3	3	3	3
Australia	0.0613	0.2352	0.7932	2	2	2	1
Austria	0.0954	0.2141	0.6918	3	3	3	2
Azerbaijan	0.1064	0.1921	0.6437	3	3	3	3
Bahamas	0.1180	0.1881	0.6145	3	3	3	3
Scenario #1: TOPSIS-Sort-B Boundary Profiles							
P_1	0.0577	0.2309	0.8	–	–	–	–
P_2	0.0866	0.2021	0.7	–	–	–	–
P_3	0.1155	0.1732	0.6	–	–	–	–
P_4	0.1443	0.1443	0.5	–	–	–	–
Scenario #2: TOPSIS-Sort-C Characteristic Profiles (considering the median of the intervals)							
P_1	0.0289	0.2598	0.9	–	–	–	–
P_2	0.0722	0.2165	0.7	–	–	–	–
P_3	0.1010	0.1876	0.65	–	–	–	–
P_4	0.1299	0.1588	0.55	–	–	–	–
P_5	0.2165	0.0722	0.25	–	–	–	–
Scenario #3: TOPSIS-Sort-C Characteristic Profiles (considering the average scores of the original allocations)							
P_1	0.0464	0.2451	0.8408	–	–	–	–
P_2	0.0710	0.2212	0.7570	–	–	–	–
P_3	0.1081	0.1893	0.6364	–	–	–	–
P_4	0.1405	0.1588	0.5306	–	–	–	–
P_5	0.1729	0.1303	0.4297	–	–	–	–

Table 9
Similarity Percentages.

Category	number of original assignments	TOPSIS-Sort-B	TOPSIS-Sort-C	
			Scenario #2	Scenario #3
C_1	6	66.67%	33.33%	66.67%
C_2	29	89.66%	79.31%	93.10%
C_3	59	89.83%	76.27%	86.44%
C_4	64	82.81%	68.75%	81.25%
C_5	22	77.27%	63.64%	81.82%
Total	180	89.44%	77.78%	90.00%

Table 10
Countries assigned differently in the three scenarios.

Country	Global Score	Standard Deviation	Original Assignment	TOPSIS Assignments
Australia	80.9	9.353817	1	2
Austria	72.0	17.5897	2	3
Finland	74.9	23.294	2	3
Ireland	80.5	6.994298	1	2
Latvia	70.4	16.76381	2	3
Macau	71.0	18.42927	2	3
Macedonia	71.1	12.34906	2	3
Norway	73.0	19.67385	2	3

In Table 8, it is important to notice how the closeness coefficients of the characteristic profiles changed between Scenario #2 and Scenario #3. The largest difference is associated with category C_5, this being 0.25 for Scenario #2 and 0.4297 for Scenario #3.

Table 9 presents the percentages of similarities in the final classification of the three scenarios tested in relation to the original assignments made by the Heritage Foundation for each category and considering all the 180 alternatives. TOPSIS-Sort-B assigned 89.44% of the alternatives to the same category as the original classification did. The highest percentage is represented by the alternatives originally assigned to C_3 (89.83% of 59 alternatives) while the lowest value was associated to the category C_1 (66.67% of 6 alternatives).

On analyzing the results in the two TOPSIS-Sort-C scenarios, we notice that C_1 was also the one with the lowest similarity. The percentage achieved in Scenario #2, represented by the median profiles of the intervals, is the lowest. This was expected since a characteristic profile should represent a typical alternative and the Heritage rule uses boundary values which sort the final evaluation of the alternatives among the categories. When analyzing the averages obtained in Table 5, we see that the countries' performances are not well represented by the medians of the intervals. On the other hand, in Scenario #3, the overall similarity percentage achieved by TOPSIS-Sort-C was 90% (162 alternatives were assigned to the original category). In this scenario, the method also achieved the highest similarity for an individual category (93.1% for C_2).

In general, the proposed MCDM/A methods succeed in classifying many alternatives in a real sorting problem. The results were coherent and differences in comparison to the original classification of Heritage Foundation were expected since instead of working with a simple average value, TOPSIS algorithms observe the distances of each alternative for an ideal and an anti-ideal situation. Thus, the compensation among criteria values occurs differently. On the other hand, it can be seen from Table 8 that all the differences in classification happened between consecutive categories, which confirms the consistency of the results.

In total, 9 countries were assigned differently in relation to the Heritage Foundation in the scenarios. Table 10 illustrates these situations by detailing the overall score for economic freedom given by the Foundation and the standard deviation over the scores of the 12 criteria. An interesting situation of compensation can be seen for Austria, which received a score of only 24.5 in the criterion of "government spending". The overall evaluation was compensated by better performances for the other criteria, as can be seen in the Decision Matrix presented in Table 6. While this compensation was enough to classify the alternative in C_2 by using the simple average of the Heritage Foundation, this did not happen with the proposed TOPSIS procedures.

In order to test the similarity between the results obtained with the

Table 11
Similarities between the Heritage Foundation (HF) method, the proposed TOPSIS-Sort methods and classic sorting methods.

Scenario	HF	TOPSIS-Sort			PROMETHEE Flow-Sort		ELECTRE-TRI			
		B	C	C	B	C	B	C	C	C
		#1	#2	#3	#4	#5	#6	#7	#8	#9
HF	100%									
#1	89.4%	100%								
#2	77.8%	88.3%	100%							
#3	**90.0%**	89.4%	86.7%	100%						
#4	54.4%	49.4%	60.6%	59.4%	100%					
#5	63.3%	61.7%	72.8%	71.7%	75.6%	100%				
#6	16.1%	17.2%	7.2%	12.8%	5.6%	12.8%	100%			
#7	37.8%	44.4%	45.6%	39.4%	17.2%	36.7%	43.9%	100%		
#8	47.2%	43.9%	51.7%	49.4%	63.9%	62.2%	8.9%	21.7%	100%	
#9	86.7%	88.9%	95.6%	91.7%	87.8%	98.3%	44.4%	100%	100%	100%

proposed TOPSIS-based methods and the results that one would obtain using classic MCDM/A sorting approaches, PROMETHEE-based and ELECTRE-based methods were applied to the same numerical example. It is important to note that different MCDM/A methods work under different paradigms and follow different guidelines. Therefore, it is not expected that different sorting methods result in the same assignments. PROMETHEE and ELECTRE methods, for example, are outranking methods and should be used when a non-compensatory relation among the criteria is expected, while TOPSIS is distance-based approach and allows compensation when the criteria are evaluated. Additional parameters may also be necessary depending on the method used for the decision process.

In addition to the three TOPSIS-based Scenarios, two variations of PROMETHEE II Flow-Sort (Nemery & Lamboray, 2008) were tested, one using boundary profiles (Scenario #4) and another one using central profiles (Scenario #5). Regarding the use of PROMETHEE, the "usual criterion" was used for this numerical example (further details about the choice of usual criteria are to be found in (Brans & Vincke, 1985)). For the ELECTRE variations, we used a cutting-level $\lambda = 0.7$ and constructed the pseudo-criteria using respectively $qt = 2$, $pt = 5$, and $v = 15$ as the indifference, preference, and veto thresholds (further information about ELECTRE parameters can be found in (de Almeida et al., 2015)). Regarding the ELECTRE family, we tested the classic ELECTRE-TRI-B (Scenario #6) in addition to ELECTRE-TRI-C. As ELECTRE-TRI-C does not always assign the alternatives to a unique class, three cases were considered for this method: the worst possible allocation (Scenario #7), the best possible allocation (Scenario #8), and the whole interval (Scenario #9). The results are summarized in Table 11 where we compare the overall similarities among the scenarios tested.

Analysis of Table 11 shows that the TOPSIS-Sort-C method in Scenario #3 was the one which achieved the greatest similarity with the original assessment of the Heritage Foundation (HF). The good similarity found for ELECTRE-TRI-C in Scenario #9 is a consequence of the imprecise results given by this method, considering an interval of possible assignments. Note also that TOPSIS-Sort-C (scenarios #2 and #3) sorting results fall into more than 90% similarity with the ELECTRE-TRI-C wide interval. Although each method brings different perspectives within its assumptions, and therefore, different results are expected, what numerical experiments can do is to let the sensitivity on assumptions be observed in the light of the expected results. Thus, one could expect that row #6 would have the smallest values of similarity since stronger outranking relations are considered to have less flexibility in its ELECTRE algorithm. Thus, the greater similarity between the Heritage Foundation and the TOPSIS-based approaches is a direct consequence of the additive properties of such methodologies, which are not found in outranking approaches.

5. Conclusion

This paper has presented TOPSIS variations for sorting problems that already include solutions that have recently been developed for the TOPSIS ranking reversal problem. TOPSIS-Sort-B is presented as an improved version of TOPSIS-Sort, as this includes a step for determining a domain for each criterion and an interval normalization option. This method is addressed to problems with boundary profiles. For the problems where characteristics profiles are used, this paper proposes TOPSIS-Sort-C.

A numerical application was made to test and validate the use of the proposed algorithms. The problem concerned the classification of 180 countries into five ordinal categories. First, the methodology used by the Heritage Foundation for its Index of Economic Freedom was analyzed, and on which basis the parameters were set for the proposed TOPSIS applications. Three scenarios were tested, and the proposed methods performed well when the results from them were matched against the original assignments of the Foundation. TOPSIS-Sort-B assigned 89.44% of the alternatives to their original categories while TOPSIS-Sort-C achieved 90%.

The results showed coherence and consistency with what was expected. For all scenarios, the differences in the classifications occurred only between consecutive categories and these are caused by the differences in the methodology of classification. Some differences were expected because while the Heritage Foundation uses a simple average to determine a global score for the alternatives, the proposed methods have the traditional characteristics of TOPSIS, and thus calculate performances based on differences from ideal and anti-ideal solutions.

Therefore, the main contributions of this paper include: a discussion regarding the impact of ranking reversal when sorting with TOPSIS; an improved version of TOPSIS for sorting that uses boundary profiles; a new method, TOPSIS-Sort-C, for sorting problems with characteristic profiles; and a new MCDM/A application which has a large set of alternatives. Regarding future studies, we expect to conduct new applications of the proposed methods to tackle different sorting problems. Furthermore, it would be useful to explore the Economic Freedom sorting problem more thoroughly. For instance, we discussed when analyzing Table 5 that the averages for two criteria seemed inconsistent with the original classification. Thus, it would be useful for future studies to investigate whether this behavior is also to be found in past years and if these criteria are necessary. To do so, statistical analysis, rule-based methods, or preference disaggregation can be applied in order to examine and reach a fuller understanding of this situation.

Author Contribution Statement

D.F de Lima Silva and A.T. de Almeida-Filho were responsible for conceptualization, methodological development and writing (original manuscript), D.F de Lima Silva was responsible for data collection, and analysis; D.F de Lima Silva and A.T. de Almeida-Filho were responsible for the manuscript revisions, D.F de Lima Silva and A.T. de Almeida-Filho were involved in the validation.

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Appendix A. - Proofs

This appendix discusses verifications on the properties observed for the TOPSIS-Sort-B and TOPSIS-Sort-C methods.

Property 1. Conformity

– Proof: The calculus of the closeness coefficient of any alternative depends on its evaluation, the weights, and the domain. Therefore, given a vector of weights $W = [w_1, \cdots, w_n]$ and domain alternatives $\{a^*, a^-\}$, if $a_{i,j} = P_{k,j}\ \forall\ j$, then it is always true that $Cl(a_i) = Cl(P_k)$.

Step 9 of TOPSIS-Sort-B takes an optimistic rule in the limit situation, allocating a_i to class C_k whenever $Cl(a_i) = Cl(P_k)$.Thus, this property is verified.

Step 9 of TOPSIS-Sort-C allocates any alternative a_i to the class represented by the closest profile in terms of the closeness coefficient. If $Cl(a_i) = Cl(P_k), |Cl(a_i) - Cl(P_k)| = 0$, which is the minimum possible distance, and alternative a_i is allocated to C_k. If Assumption 4 is true, no other profile will have the same closeness coefficient as a_i.

Property 2. Homogeneity

– Proof: If two alternatives a_i and a_s have the same distance from the ideal and anti-ideal solutions, $d^*_{a_i} = d^*_{a_s}$ and $d^-_{a_i} = d^-_{a_s}$. From Step 8 of Algorithms 3 and 4, the closeness coefficient of any alternative depends only on its distances from the ideal and the anti-ideal solution Therefore: $Cl(a_i) = Cl(a_s)$. From Step 9 of Algorithms 3 and 4, we see that the classification of each alternative depends only on its closeness coefficient and the closeness coefficient of the profiles. Therefore, it is impossible that $Cl(a_i) = Cl(a_s)$ and they are allocated to different classes and Property 2 is verified for both methods.

Property 3. Monotonicity

– Proof: If alternative a_i dominates alternative a_s then $a_{i,j} \succcurlyeq a_{s,j}\ \forall\ j$ and $a_{i,j} \succ a_{s,j}$ for at least one criterion. Then, after the normalization steps and the determination of the ideal and anti-ideal solutions, v_i is closer to v^* and farther from v^- in comparison to v_s. In other words, $d^*_{a_i} < d^*_{a_s}$ and $d^-_{a_i} > d^-_{a_s}$. Therefore, $\frac{d^-_{a_i}}{d^*_{a_i} + d^-_{a_i}} > \frac{d^-_{a_s}}{d^*_{a_s} + d^-_{a_s}}$ which means that $Cl(a_i) > Cl(a_s)$. Using the same logic, if Assumptions 1 and 4 are true, we know the closeness coefficients of the profiles are ordered: $Cl(P_1) > Cl(P_2) > \cdots > Cl(P_p)$.

In Step 9 of TOPSIS-Sort-B, assuming $Cl(a_i) > Cl(a_s)$, it is impossible that $Cl(a_s) \geq Cl(P_k)$ and $Cl(a_i) < Cl(P_k)$. Thus, a_i is allocated to a class at least as good as a_s and Property 3 is verified.

In Step 9 of TOPSIS-Sort-C, assuming $Cl(a_i) > Cl(a_s)$, if $Cl(P_k)$ is the closest closeness coefficient to $Cl(a_i)$ and $Cl(P_t)$ is the closest closeness coefficient to $Cl(a_s)$, it is impossible that $k > t$. In other words, a_i is allocated to a class at least as good as a_s and Property 3 is verified.

Property 4. Irreversibility (ranking reversal protection)

– Proof: We know from Step 9 of TOPSIS-Sort-B and TOPSIS-Sort-C that the allocation of any alternative a_i depends on comparisons between its closeness coefficient $Cl(a_i)$ and the closeness coefficients of profiles $Cl(P_k)$, where $k = 1, 2 \cdots p$. In addition, we know from Step 7 and Step 8 that the calculation of the closeness coefficients is based on distances from the corresponding alternative/profile and the ideal v^* and anti-ideal v^- solutions. From Step 6, we know the ideal and anti-ideal solutions are obtained using maximum and minimum values from the normalized decision matrix $V = [v_{i,j}]_{(m+p+2)\times n}$.

$$v^* = [v_1^*, v_2^*, \cdots, v_n^*],\ v_j^* = \begin{cases} \max_i v_{i,j},\ g_j \in G^+ \\ \min_i v_{i,j},\ g_j \in G^- \end{cases}$$

$$v^- = [v_1^-, v_2^-, \cdots, v_n^-],\ v_j^- = \begin{cases} \min_i v_{i,j},\ g_j \in G^+ \\ \max_i v_{i,j},\ g_j \in G^- \end{cases}$$

If assumption 2 is true, the domain of the criteria is known. Moreover, Step 3 from TOPSIS-Sort-B and TOPSIS-Sort-C defines alternatives that describe this domain (a^* and a^-) and these alternatives are included in the initial decision matrix. The definition of these alternatives guarantees that the construction of the ideal and anti-ideal solutions will not change after the inclusion or elimination of alternatives from the initial set $A = \{a_1, a_2, \cdots, a_m\}$. This happens because the modification of the initial set of alternatives will not expand or compress the domain of the criteria. In other words, as well as the initial set A, the new set of alternatives will respect the relation $a_j^* \geq a_{i,j} \geq a_j^-,\ \forall\ i, j$. Therefore, changes in the initial set of alternatives do not impact the calculation of closeness coefficients of the profiles and the remaining alternatives. As a result, the initial class allocation does not change, and Property 4 is verified.

Property 5. Strong Stability

– Proof: Because of the dominance condition of the profiles described in Assumption 4 and the monotonicity property (Property 3) we know that $Cl(P_1) > \cdots > Cl(P_{k-1}) > Cl(P_k) > Cl(P_{k+1}) > \cdots > Cl(P_p)$. In addition, we know the operations described in Definition 1 and Definition 2 respect Assumption 4 and Property 3.

Consider a merging operation between C_k and C_{k+1}, where a new class $C_k^{'}$ is constructed. After the merging operation: $Cl(P_{k-1}) > Cl(P_k^{'}) > Cl(P_{k+2})$. Thus:

i. If alternative a_i is initially allocated to class C_k, then, before the operation: $Cl(P_{k-1}) > Cl(a_i) \geq Cl(P_k)$. After the operation, $Cl(P_{k-1}) > Cl(a_i) \geq Cl(P_k^{'})$ and the alternative is allocated to the new class $C_k^{'}$.
ii. If alternative a_i is initially allocated to class C_{k+1}, then: $Cl(P_{k-1}) > Cl(P_k) > Cl(a_i) \geq Cl(P_{k+1})$. After the operation, $Cl(P_{k-1}) > Cl(a_i) \geq Cl(P_k^{'})$ and the alternative is allocated to the new class $C_k^{'}$.

iii. If alternative a_i is initially allocated to class C_{k-1} or better, then: $Cl(a_i) > Cl(P_k)$. After the operation, $Cl(a_i) > Cl(P_k^{'})$ and the alternative remains in the same class as the closeness coefficients of the other profiles do not change.
iv. If alternative a_i is initially allocated to class C_{k+2} or worse, then: $Cl(P_{k+1}) > Cl(a_i)$. After the operation, $Cl(P_k^{'}) > Cl(a_i)$ and the alternative remains in the same class as the closeness coefficients of the other profiles do not change.

Considering that class C_k is split between the classes $C_k^{'}$ and $C_k^{''}$,where $C_k^{'} \succ C_k^{''}$, a new profile is created $P_k^{'}$, defined in the limit between the two new classes. The following relation is respected: $Cl(P_{k-1}) > Cl(P_k^{'}) > Cl(P_k)$. Thus:

i. If alternative a_i is initially allocated to class C_k, then, before the operation: $Cl(P_{k-1}) > Cl(a_i) \geq Cl(P_k) > Cl(P_{k+1})$. After the operation, we know that $Cl(a_i) \geq Cl(P_k)$. Therefore, if $Cl(a_i) \geq Cl(P_k^{'})$, alternative a_i is allocated to the new class $C_k^{'}$. Otherwise, alternative a_i is allocated to the new class $C_k^{''}$.
ii. If alternative a_i is initially allocated to class C_{k-1} or better, then: $Cl(a_i) \geq Cl(P_{k-1})$ and these closeness coefficients do not change after the split operation. As after the operation $Cl(P_{k-1}) > Cl(P_k^{'}) > Cl(P_k)$ because of the monotonicity property, and the closeness coefficients of the other profiles do not change, the alternative remains in the same class.
iii. If alternative a_i is initially allocated to class C_{k+1} or worse, then: $Cl(P_k) > Cl(a_i)$ and these closeness coefficients do not change after the split operation. As after the operation $Cl(P_k^{'}) > Cl(P_k) > Cl(a_i)$ because of the monotonicity property, and the closeness coefficients of the other profiles do not change, the alternative remains in the same class.

Property 6. Stability

– Proof: As Property 5 is a more constrained version of Property 6, the Stability condition is already proved for TOPSIS-Sort-B. In TOPSIS-Sort-C, the classification of alternatives occurs based on the distance of their closeness coefficients to the closeness coefficients of the profiles. Because of the dominance condition of profiles described in Assumption 4 and the monotonicity property (Property 3) we know that $Cl(P_1) > \cdots > Cl(P_{k-1}) > Cl(P_k) > Cl(P_{k+1}) > \cdots > Cl(P_p)$. In addition, we know the operations described in Definition 1 and Definition 2 respect Assumption 4 and Property 3.

Considering classes C_k and C_{k+1} are merged, creating a new class $C_k^{'}$. The adjacent classes to the modified ones are C_{k-1} and C_{k+2}. After the merging operation, the closeness coefficient of the new profile $P_k^{'}$ must be larger than $Cl(P_{k+1})$ and smaller than $Cl(P_k)$, then: $Cl(P_k) > Cl(P_k^{'}) > Cl(P_{k+1})$.

i. If an alternative a_i is allocated initially to a non-adjacent class, it belongs to class C_{k-2} (or better) or to class C_{k+3} (or worse).
   a. In the first case, we know that $|Cl(a_i) - Cl(P_{k-2})| \leq |Cl(a_i) - Cl(P_{k-1})| < |Cl(a_i) - Cl(P_k)|$. After the merger, this relation does not change, and the new closeness coefficient will be even farther from $Cl(a_i)$ because $Cl(a_i) > Cl(P_k) > Cl(P_k^{'})$.
   b. In the second case, the following relation would be true before the operation: $|Cl(a_i) - Cl(P_{k+3})| < |Cl(a_i) - Cl(P_{k+2})| \leq |Cl(a_i) - Cl(P_{k+1})|$. Again, the merge operation would not affect this relation and the new closeness coefficient would be even farther from $Cl(a_i)$ because $Cl(P_k^{'}) > Cl(P_{k+1}) > Cl(a_i)$.
ii. Considering an initial allocation to a class adjacent to the modified one, then a_i belongs to C_{k-1} or C_{k+2} before the operation.
   a. If a_i is initially allocated to C_{k-1}, we know that $|Cl(a_i) - Cl(P_{k-1})| \leq |Cl(a_i) - Cl(P_k)|$ and as a consequence $Cl(a_i) > Cl(P_k)$. As after the operation $Cl(P_k) > Cl(P_k^{'})$, we know that $|Cl(a_i) - Cl(P_{k-1})| < |Cl(a_i) - Cl(P_k^{'})|$ and the alternative is allocated to the same class.
   b. If a_i is initially allocated to C_{k+2}, then $|Cl(a_i) - Cl(P_{k+2})| < |Cl(a_i) - Cl(P_{k+1})|$ and $Cl(a_i) < Cl(P_{k+1})$. As after the operation $Cl(P_k^{'}) > Cl(P_{k+1})$, then $|Cl(a_i) - Cl(P_{k+2})| < |Cl(a_i) - Cl(P_k^{'})|$ and the alternative is allocated to the same class. The principle is the same as that used for non-adjacent classes.
iii. Considering an initial allocation to a merged class, then a_i initially belongs to C_k or C_{k+1}.
   a. If a_i is initially allocated to C_k, we know that $|Cl(a_i) - Cl(P_k)| < |Cl(a_i) - Cl(P_{k-1})|$, and so $Cl(P_{k-1}) > Cl(a_i)$. In addition, we know that $|Cl(a_i) - Cl(P_k)| \leq |Cl(a_i) - Cl(P_{k+1})|$, and so $Cl(a_i) > Cl(P_{k+1})$. Therefore, or $Cl(P_{k-1}) > Cl(a_i) \geq Cl(P_k) > Cl(P_{k+1})$ or $Cl(P_{k-1}) > Cl(P_k) \geq Cl(a_i) > Cl(P_{k+1})$. After merging to C_k and C_{k+1} and creating profile $P_k^{'}$, we know that $Cl(P_k) > Cl(P_k^{'}) > Cl(P_{k+1})$. Thus, two allocations are possible. If $|Cl(a_i) - Cl(P_k^{'})| < |Cl(a_i) - Cl(P_{k-1})|$, then, the alternative is allocated to the new class: $C_k^{'}$. Otherwise, a_i is allocated to the adjacent class C_{k-1}.
   b. If a_i is initially allocated to C_{k+1}, we know that $|Cl(a_i) - Cl(P_{k+1})| < |Cl(a_i) - Cl(P_k)|$, and so $Cl(P_k) > Cl(a_i)$. In addition, we know that $|Cl(a_i) - Cl(P_{k+1})| \leq |Cl(a_i) - Cl(P_{k+2})|$, and so $Cl(a_i) > Cl(P_{k+2})$. Therefore, either $Cl(P_k) > Cl(a_i) \geq Cl(P_{k+1}) > Cl(P_{k+2})$ or $Cl(P_k) > Cl(P_{k+1}) \geq Cl(a_i) > Cl(P_{k+2})$. After merging to C_k and C_{k+1} and creating profile $P_k^{'}$, we know that $Cl(P_k) > Cl(P_k^{'}) > Cl(P_{k+1})$. Thus, two allocations are possible. If $|Cl(a_i) - Cl(P_k^{'})| \leq |Cl(a_i) - Cl(P_{k+2})|$, then, the alternative is allocated to the new class: $C_k^{'}$. Otherwise, a_i is allocated to the adjacent class C_{k+2}.

Considering that class C_k is split into two classes $C_k^{'}$ and $C_k^{''}$. Then, profile P_k is also split and $Cl(P_k^{'}) > Cl(P_k) > Cl(P_k^{''})$.

i. If an alternative a_i is allocated initially to a non-adjacent class, it belongs to class C_{k-2} (or better) or to class C_{k+2} (or worse).
   a. In the first case, we know that $|Cl(a_i) - Cl(P_{k-2})| \leq |Cl(a_i) - Cl(P_{k-1})| < |Cl(a_i) - Cl(P_k)|$, and so $Cl(a_i) > Cl(P_{k-1})$. After the splitting operation, we know that $Cl(P_{k-1}) > Cl(P_k^{'}) > Cl(P_k^{''})$ because of the monotonicity property. Thus, as the closeness coefficients of the non-modified profiles do not change after the operation, the initial allocation of the alternative will not be affected, and the alternative remains allocated to the same class.
   b. In the second case, we know that $|Cl(a_i) - Cl(P_{k+2})| < |Cl(a_i) - Cl(P_{k+1})| < |Cl(a_i) - Cl(P_k)|$, and so $Cl(P_{k+1}) > Cl(a_i)$. After the splitting operation, we know that $Cl(P_k^{'}) > Cl(P_k^{''}) > Cl(P_{k+1})$ because of the monotonicity property. Thus, as the closeness coefficients of the non-modified profiles do not change after the operation, the initial allocation of the alternative will not be affected, and the alternative remains allocated to the same class.
ii. If an alternative a_i is allocated initially to an adjacent class, it belongs to class C_{k-1} or to class C_{k+1}.
   a. In the first case, we know that $|Cl(a_i) - Cl(P_{k-1})| \leq |Cl(a_i) - Cl(P_k)|$, and so $Cl(a_i) > Cl(P_k)$. As $Cl(P_k^{'}) > Cl(P_k)$, after the operation two allocations are possible. If $|Cl(a_i) - Cl(P_{k-1})| \leq |Cl(a_i) - Cl(P_k^{'})|$, alternative a_i remains in the same class, otherwise, the alternative is allocated

to the new class $C_k^{'}$.

b. In the second case, we know that $|Cl(a_i) - Cl(P_{k+1})| < |Cl(a_i) - Cl(P_k)|$, and so $Cl(P_k) > Cl(a_i)$. As $Cl(P_k^{''}) < Cl(P_k)$, after the operation two allocations are possible. If $|Cl(a_i) - Cl(P_k^{''})| \leq |Cl(a_i) - Cl(P_{k+1})|$, alternative a_i is allocated to the new class $C_k^{''}$, otherwise the alternative remains allocated in the same class.

iii. Considering that alternative a_i is initially allocated to the modified class C_k. Then, it is true that $|Cl(a_i) - Cl(P_k)| < |Cl(a_i) - Cl(P_{k-1})|$, and so $Cl(P_{k-1}) > Cl(a_i)$. It is also true that $|Cl(a_i) - Cl(P_k)| \leq |Cl(a_i) - Cl(P_{k+1})|$, and so $Cl(a_i) > Cl(P_{k+1})$. Therefore, either $Cl(P_{k-1}) > Cl(P_k) \geq Cl(a_i) > Cl(P_{k+1})$ or $Cl(P_{k-1}) > Cl(a_i) \geq Cl(P_k) > Cl(P_{k+1})$, or both are true in case $Cl(P_k) = Cl(a_i)$. We know that after the split, the two new profiles will respect the relation $Cl(P_{k-1}) > Cl(P_k^{'}) > Cl(P_k) > Cl(P_k^{''}) > Cl(P_{k+1})$. Thus, it is impossible that $|Cl(a_i) - Cl(P_{k-1})| \leq |Cl(a_i) - Cl(P_k^{'})|$ and it is also impossible that $|Cl(a_i) - Cl(P_{k+1})| < |Cl(a_i) - Cl(P_k^{''})|$. Therefore, as class C_k will no longer exist, two allocations are possible. If $|Cl(a_i) - Cl(P_k^{'})| \leq |Cl(a_i) - Cl(P_k^{''})|$, the alternative is allocated to the new class $P_k^{'}$. Otherwise, it is allocated to the new class $P_k^{''}$.

Appendix B. Supplementary material

Supplementary data to this article can be found online at https://doi.org/10.1016/j.cie.2020.106328.

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