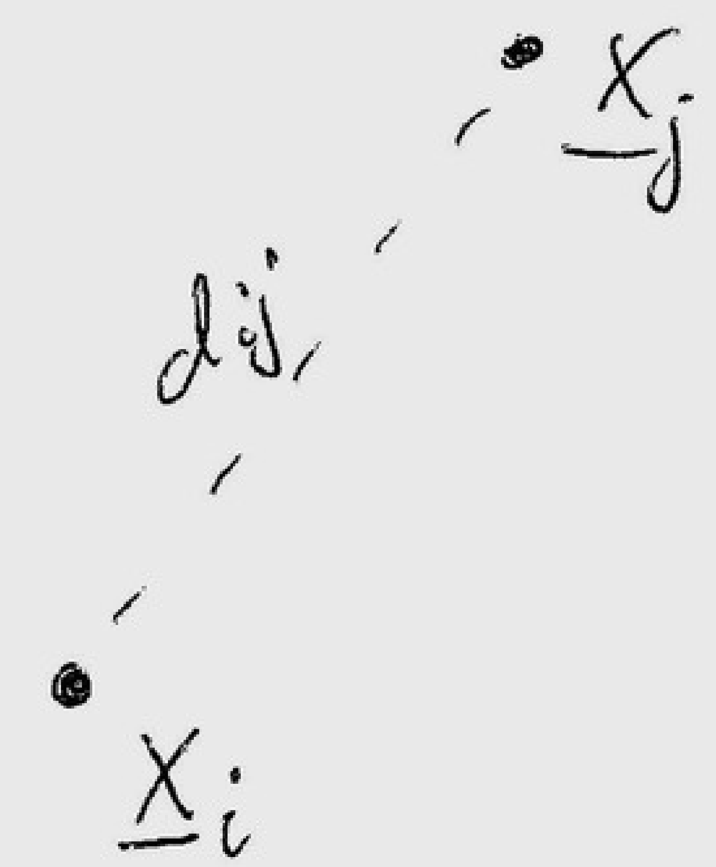
$$d_{ij}^2 = b_{ii} + b_{jj} - 2b_{ij} \qquad X = [m \times n]$$

n – wymiarowa przestrzeń
m – wektorów

ad (2)

$$B = X \cdot X^T = [m \times n] \times [n \times m] = [m \times m]$$

$$= X \cdot X^T = \begin{bmatrix} \boxed{\underline{x}_1^T} \\ \boxed{\underline{x}_2^T} \\ \vdots \\ \boxed{\underline{x}_m^T} \end{bmatrix} \times \begin{bmatrix} \boxed{\underline{x}_1} & \boxed{\underline{x}_2} & \cdots & \boxed{\underline{x}_m} \end{bmatrix} = \begin{bmatrix} \underline{x}_1^T \cdot \underline{x}_1 & \underline{x}_1^T \cdot \underline{x}_2 & \cdots & \underline{x}_1^T \cdot \underline{x}_m \\ \underline{x}_2^T \cdot \underline{x}_1 & \underline{x}_2^T \cdot \underline{x}_2 & \cdots & \underline{x}_2^T \cdot \underline{x}_m \\ \underline{x}_m^T \cdot \underline{x}_1 & \underline{x}_m^T \cdot \underline{x}_2 & \cdots & \underline{x}_m^T \cdot \underline{x}_m \end{bmatrix}$$



$$\underline{x}_i^T = [x_{i1}, x_{i2}, \ldots, x_{in}]$$

$$\underline{x}_j^T = [x_{j1}, x_{j2}, \ldots, x_{jn}]$$

$i \neq j$

$$d_{ij}^2 = (x_{i1} - x_{j1})^2 + (x_{i2} - x_{j2})^2 + \cdots + (x_{in} - x_{jn})^2$$

$$b_{ii} + b_{jj} - 2b_{ij} = \underline{x}_i^T \cdot \underline{x}_i + \underline{x}_j^T \cdot \underline{x}_j - 2\,\underline{x}_i^T \cdot \underline{x}_j = x_{i1} \cdot x_{i1} + x_{i2} \cdot x_{i2} + \cdots + x_{in} \cdot x_{in} +$$
$$x_{j1} \cdot x_{j1} + x_{j2} \cdot x_{j2} + \cdots + x_{jn} \cdot x_{jn} +$$
$$-2(x_{i1} \cdot x_{j1} + x_{i2} \cdot x_{j2} + \cdots + x_{in} \cdot x_{jn}) =$$

$$= x_{i1}^2 + x_{i2}^2 + \cdots + x_{in}^2 +$$
$$x_{j1}^2 + x_{j2}^2 + \cdots + x_{jn}^2 +$$
$$-2x_{i1}x_{j1} - 2x_{i2}x_{j2} - \cdots - 2x_{in}x_{jn} = (x_{i1} - x_{j1})^2 + (x_{i2} - x_{j2})^2 + \cdots + (x_{in} - x_{jn})^2 = d_{ij}^2$$

$\square$

ad 3)

$$B = X \cdot X^T = \begin{bmatrix} \boxed{\quad x_1^T \quad} \\ \boxed{\quad x_2^T \quad} \\ \vdots \\ \boxed{\quad x_m^T \quad} \end{bmatrix} \times \begin{bmatrix} \boxed{x_1} & \boxed{x_2} & \cdots & \boxed{x_m} \end{bmatrix} = \begin{bmatrix} x_1^T \cdot x_1 & x_1^T \cdot x_2 & \cdots & x_1^T \cdot x_m \\ x_2^T \cdot x_1 & x_2^T \cdot x_2 & \cdots & x_2^T \cdot x_m \\ \vdots & & & \\ x_m^T \cdot x_1 & x_m^T \cdot x_2 & \cdots & x_m^T \cdot x_m \end{bmatrix}$$

$$\sum_{j=1}^{m} b_{ij} = 0$$ – suma w wierszu i-tym = 0

$$\sum_{j=1}^{m} b_{ij} = x_i^T \cdot x_1 + x_i^T \cdot x_2 + \cdots + x_i^T \cdot x_m = x_{i1} \cdot x_{11} + x_{i2} \cdot x_{12} + \cdots + x_{in} \cdot x_{1n} +$$

$$+ x_{i1} \cdot x_{21} + x_{i2} \cdot x_{22} + \cdots + x_{in} \cdot x_{2n} +$$

$$+ \cdots +$$

$$+ x_{i1} \cdot x_{m1} + x_{i2} \cdot x_{m2} + \cdots + x_{in} \cdot x_{mn} =$$

$$= x_{i1} \overbrace{(x_{11} + x_{21} + \cdots + x_{m1})}^{\text{suma w kol. } 1 = 0} + x_{i2} \overbrace{(x_{12} + x_{22} + \cdots + x_{m2})}^{\text{suma w kol. } 2 = 0} + \cdots + x_{in} \overbrace{(x_{1n} + x_{2n} + \cdots + x_{mn})}^{\text{suma w kol. } n = 0} = 0$$

$\square$

ad 3)

$$\sum_{i=1}^{m} b_{ij} = 0$$ – suma w kolumnie j-tej = 0

$$\sum_{i=1}^{m} b_{ij} = X_1^T \cdot X_j + X_2^T \cdot X_j + \ldots + X_m^T \cdot X_j = X_j^T \cdot X_1 + X_j^T \cdot X_2 + \ldots + X_j^T \cdot X_m = 0$$

(patrz str. 2)

macierz $B = X \cdot X^T$ jest symetryczna!

ad 4) $$\sum_{i=1}^{m} d_{ij}^2 = tr(B) + m \cdot b_{jj}$$

$$L = \sum_{i=1}^{m} d_{ij}^2 = \sum_{i=1}^{m} b_{ii} + b_{jj} - 2b_{ij} = \sum_{i=1}^{m} b_{ii} + \sum_{i=1}^{m} b_{jj} - 2\cancelto{0}{\sum_{i=1}^{m} b_{ij}} = TR(B) + m \cdot b_{jj} = P$$

$\square$

ad 5)

$$\sum_{j=1}^{m} d_{ij}^2 = tr(B) + m \cdot b_{ii}$$

$$L = \sum_{j=1}^{m} d_{ij}^2 = \sum_{j=1}^{m} b_{ii} + b_{jj} - 2b_{ij} = \sum_{j=1}^{m} b_{ii} + \sum_{j=1}^{m} b_{jj} - 2\cancelto{0}{\sum_{j=1}^{m} b_{ij}} = TR(B) + m \cdot \overset{b_{ii}}{\cancel{b_{jj}}} = P$$

$\square$

ad 6) $$\sum_{i=1}^{m} \sum_{j=1}^{m} d_{ij}^2 = 2 \cdot m \cdot tr(B)$$

$$L = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} b_{ii} + b_{jj} - 2b_{ij} \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} b_{ii} + \sum_{j=1}^{m} b_{jj} - 2\cancelto{0}{\sum_{j=1}^{m} b_{ij}} \right) =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ii} + \sum_{i=1}^{m} \sum_{j=1}^{m} b_{jj} = \sum_{j=1}^{m} tr(B) + \sum_{i=1}^{m} tr(B) = m \cdot tr(B) + m \cdot tr(B) =$$

$$= 2 \cdot m \cdot tr(B) = P$$

$\square$

ad 7) $b_{ij} = -\frac{1}{2}\left(d_{ij}^2 - \frac{1}{m}\sum_j d_{ij}^2 - \frac{1}{m}\sum_i d_{ij}^2 + \frac{1}{m^2}\sum_{ij} d_{ij}^2\right)$

z 6): $tr(B) = \frac{1}{2m}\cdot\sum_{ij} d_{ij}^2$

Podstawiam tr(B) do wzorów 4) i 5):

4): $\sum_i d_{ij}^2 = \frac{1}{2m}\sum_{ij} d_{ij}^2 + m\cdot b_{jj} \longrightarrow b_{jj} = \frac{\sum_i d_{ij}^2 - \frac{1}{2m}\sum_{ij} d_{ij}^2}{m}$

5): $\sum_j d_{ij}^2 = \frac{1}{2m}\sum_{ij} d_{ij}^2 + m\cdot b_{ii} \longrightarrow b_{ii} = \frac{\sum_j d_{ij}^2 - \frac{1}{2m}\sum_{ij} d_{ij}^2}{m}$

z 2) mamy: $L = b_{ij} = \frac{b_{ii} + b_{jj} - d_{ij}^2}{2} = \frac{\frac{\sum_j d_{ij}^2 - \frac{1}{2m}\sum_{ij} d_{ij}^2}{m} + \frac{\sum_i d_{ij}^2 - \frac{1}{2m}\sum_{ij} d_{ij}^2}{m} - d_{ij}^2}{2} =$

$$= \frac{\sum_j d_{ij}^2 - \frac{1}{2m}\sum_{ij} d_{ij}^2 + \sum_i d_{ij}^2 - \frac{1}{2m}\sum_{ij} d_{ij}^2 - m\cdot d_{ij}^2}{2m} = \frac{\sum_j d_{ij}^2 - \frac{1}{m}\sum_{ij} d_{ij}^2 + \sum_i d_{ij}^2 - m\cdot d_{ij}^2}{2m} =$$

$$= -\frac{1}{2}\left(\frac{\cancel{m}\cdot d_{ij}^2}{\cancel{m}} - \frac{1}{m}\sum_j d_{ij}^2 - \frac{1}{m}\sum_i d_{ij}^2 + \frac{\frac{1}{m}\sum_{ij} d_{ij}^2}{m}\right) =$$

$$= -\frac{1}{2}\left(d_{ij}^2 - \frac{1}{m}\sum_j d_{ij}^2 - \frac{1}{m}\sum_i d_{ij}^2 + \frac{1}{m^2}\sum_{ij} d_{ij}^2\right) = P$$

□