Group decision making by voting

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Group decision making by voting

- In democracy most decisions are made in groups or by the community
- Voting is a possible way to make the decisions
  - Allows large number of decision makers
  - All DMs are not necessarily satisfied with the result
- The size of the group doesn’t guarantee the quality of the decision
- Competence and expertise are not always taken into account (one person = one vote)

Voting - a social choice

- n alternatives $x_1, x_2, ..., x_n$
- k voters – decision makers $DM_1, DM_2, ..., DM_k$
- each DM has preferences for the alternatives
- which alternative the group should choose?

Social choice rule - SCR

- Preference of a single voter is expressed as a ranking of alternatives (the ranking may not be complete), e.g., the preference profile:
  - DM1: A > B > C
  - DM2: B > C > A
  - DM3: C > B > A
- Social choice rule (SCR) aggregates the preference profiles into a social outcome, i.e., ranking indicating the winner (ties allowed)
- Examples: political and corporate elections, selection of employees, selection of projects, competition for grants, family vote for vacation, etc.
- SCR is imposing a voting rule

Plurality rule

- Plurality rule : each voter has one vote; the alternative that was ranked first by the greatest number of voters is the winner:

  3: A > B > C

  1: A > C > B

  3: B > C > A

  2: C > B > A

  Decision: 4 for A, 3 for B, 2 for C – A is the winner

- This is the only rule that is:
  - anonimous – each vote has the same value,
  - neutral – labels of alternatives do not influence the ranking,
  - monotonic – if a voter improves the rank of alternative x, which is a winner, then x remains the winner

- Examples: Great Britain, USA, Kanada, Kenia, Iran, Kuweit, Nepal, Singapore, South Korea, ... – 40 countries in total

Antiplurality rule and approval voting

- Antiplurality rule : each but the last alternative in individual rankings is awarded:

  3: A > B > C (the ranking may not be complete)

  1: A > C

  3: B > C > A

  2: C > B > A

  Decision: 4 for A, 8 for B, 5 for C – B is the winner

- Approval voting: each voter votes for a subset of alternatives; each alternative from a given subset gets one point; the alternative with the greatest number of points is the winner:

  4: A

  3: B, C

  2: C

  Decision: 4 for A, 3 for B, 5 for C – C is the winner

Examples: conclave (1294-1621), general secretary of UN

Antiplurality rule and approval voting

- Antiplurality rule : each but the last alternative in individual rankings is awarded:

  3: A > B > C (the ranking may not be complete)

  1: A > C

  3: B > C > A

  2: C > B > A

  Decision: 4 for A, 8 for B, 5 for C – B is the winner

- Approval voting: each voter votes for a subset of alternatives; each alternative from a given subset gets one point; the alternative with the greatest number of points is the winner:

	DM_1	DM_2	DM_3	DM_4	DM_5	DM_6	DM_7	DM_8	DM_9	total	
A	X	-	-	X	-	X	-	X	-	4	
B	X	X	X	X	X	X	-	X	-	7	the winner
C	-	-	-	-	-	-	X	-	X	2	

Examples: conclave (1294-1621), general secretary of UN

Run-off election

- Plurality run-off : the winner must get over 50% of the votes; if the condition is not met, keep only two best alternatives and repeat the voting:

  4: A > B > C

  3: B > C > A

  2: C > B > A

  Decision: 4 for A, 3 for B, 2 for C – none got 50%, keep A, B

  4: A > B

  3: B > A

  2: B > A

  Decision: 4 for A, 5 for B – B is the winner

- Examples: presidential elections in Poland, France, Brazil, Portugal, Ukraine, ...

Run-off election

- Single transferable vote: the winner must get over 50% of the votes; if the condition is not met, eliminate one alternative with the lowest number of votes and repeat the voting; continue until conclusion:

  5: A > B > C > D
  7: B > D > C > A
  7: C > B > A > D
  4: D > C > B > A

  Stage 1: 5 for A, 7 for B, 7 for C, 4 for D – none got 50%, remove D

  5: A > B > C
  7: B > C > A
  7: C > B > A
  4: C > B > A

  Stage 2: 5 for A, 7 for B, 11 for C – none got 50%, remove A

  5: B > C
  7: B > C
  7: C > B
  4: C > B

  Stage 3: 12 for B, 11 for C – B is the winner

- Examples: presidential election in Australia and New Zealand

Some paradoxes (1/2)

- Winner-turns-loser paradox: the winner may become loser if some voters increase its rank:

  27: A > B > C

  42: C > A > B

  24: B > C > A

  Plurality run-off: in stage 1, keep A and C, then C beats A 66:27

  Assume that 4 voters improved the rank of C from 3rd to 1st:

  23: A > B > C

  46: C > A > B

  24: B > C > A

  Plurality run-off: in stage 1, keep B and C, then B beats C 47:46 even if C got an additional support

Some paradoxes (2/2)

- No-show paradox: alternative that did not win until now, becomes the winner after adding additional votes where it is ranked the last:

  23: A > B > C

  46: C > A > B

  24: B > C > A

  Plurality run-off: in stage 1, keep B and C, then B beats C 47:46

  Assume that 42 additional voters vote: A > B > C

  65: A > B > C

  46: C > A > B

  24: B > C > A

  Plurality run-off: in stage 1, keep A and C, then C beats A 70:65 even if C was ranked the last in 42 additional votes

Jean Condorcet (1743-1794) – Condorcet rule

- Each pair of alternatives is compared
- The alternative which is the best in all comparisons is the winner
- There may be no solution

Consider alternatives A, B, C, 33 voters and the following voting result

	A	B	C
A	-	18,15	18,15
B	15,18	-	32,1
C	15,18	1,32	-

- A is better than B by 18:15, and better than C by 18:15

  $\Rightarrow$ A is the Condorcet winner
- Similarly, C is the Condorcet loser

Jean Condorcet (1743-1794) – Condorcet rule

- Example 1:

  1: B > C > A > D

  1: D > A > C > B

  1: A > C > B > D

vs.	A	B	C	D
A	-	2,1	2,1	2,1
B	1,2	-	1,2	2,1
C	1,2	2,1	-	2,1
D	1,2	1,2	1,2	-

A is the winner
D is the loser

- Example 2:

  1: B > C > D > A

  1: D > A > C > B

  1: A > C > B > D

vs.	A	B	C	D
A	-	2,1	2,1	1,2
B	1,2	-	1,2	2,1
C	1,2	2,1	-	2,1
D	2,1	1,2	1,2	-

There is no Condorcet winner

The Condorcet paradox

- Consider the following comparison of the three alternatives

1: A > B > C

1: B > C > A

1: C > A > B

Every alternative has a supporter!

vs.	A	B	C
A	-	2,1	1,2
B	1,2	-	2,1
C	2,1	1,2	-

Paired comparisons:

- A is preferred to B (2-1)
- B is preferred to C (2-1)
- C is preferred to A (2-1)

- The paired comparisons are cycling: A > B > C > A

Escaping the Condorcet paradox

- Pairwise voting in a given order:

1) (A-B) $\Rightarrow$ A wins, (A-C) $\Rightarrow$ C is the winner

2) (B-C) $\Rightarrow$ B wins, (B-A) $\Rightarrow$ A is the winner

3) (A-C) $\Rightarrow$ C wins, (C-B) $\Rightarrow$ B is the winner

	DM_1	DM_2	DM_3
A	1	3	2
B	2	1	3
C	3	2	1

vs.	A	B	C
A	-	2,1	1,2
B	1,2	-	2,1
C	2,1	1,2	-

The voting result depends on the pairing order

Strategic voting in case of known voting order

- DM_1 knows the preferences of the other voters and the voting order (A-B, B-C, A-C)
- The favourite A of DM_1 cannot win*
- If DM_1 votes for B instead of A in the first round
  - B is the winner
  - DM_1 avoids the least preferred alternative C

	DM_1	DM_2	DM_3
A	1	3	2
B	2	1	3
C	3	2	1

	DM_1	DM_2	DM_3
A	2	3	2
B	1	1	3
C	3	2	1

vs.	A	B	C
A	-	2,1	1,2
B	1,2	-	2,1
C	2,1	1,2	-

→

vs.	A	B	C
A	-	1,2	1,2
B	2,1	-	2,1
C	2,1	1,2	-

* If DM_2 and DM_3 vote according to their preferences

In case there is no Condorcet winner

- Copeland rule : the alternative for which the difference between the number of won and the number of lost pairwise comparisons with other alternatives is the greatest, is the winner:

31 : A > E > C > D > B

30 : B > A > E > C > D

29 : C > D > B > A > E

10 : D > A > B > C > E

vs.	A	B	C	D	E
A	-	41,59	71,29	61,39	100,1
B	59,41	-	40,60	30,70	69,31
C	29,71	60,40	-	90,10	39,61
D	39,61	70,30	10,90	-	39,61
E	0,100	31,69	61,39	61,39	-

- Decision: A (won 3, lost 1), B (2 vs. 2), C (2 vs. 2), D (1 vs. 3), E (2 vs. 2) – Copeland winner: A

In case there is no Condorcet winner

- Kemeny rule : among all permutations, choose the ranking being the closest to the voters' profiles, i.e. maximizing the total number of concordant pairwise comparisons:

  7 : M > W > B

  9 : W > B > M

  4 : B > M > W

vs.	M	W	B
M	-	11	7
W	9	-	16
B	13	4	-

  Kemeny number of concordant pairwise comparisons:

  All permutations

  M W B : (M vs. W = 11) + (M vs. B = 7) + (W vs. B = 16) = 34

  M B W : (M vs. B = 7) + (M vs. W = 11) + (B vs. W = 4) = 22

  W M B : (W vs. M = 9) + (W vs. M = 16) + (M vs. B = 7) = 32

  W B M : (W vs. B = 16) + (W vs. M = 9) + (B vs. M = 13) = 38

  B M W : (B vs. M = 13) + (B vs. W = 4) + (M vs. W = 11) = 28

  B W M : (B vs. W = 4) + (B vs. M = 13) + (W vs. M = 9) = 26

- Decision: W > B > M

In case there is no Condorcet winner

- Maxmin rule : rank the alternatives in the order of decreasing minimum numbers of pairwise comparisons being won by them:

7 : M > W > B

9 : W > B > M

4 : B > M > W

vs.	M	W	B	min won
M	-	11	7	7
W	9	-	16	9
B	13	4	-	4

- Let score(X,Y) be the number of voters who prefer X over Y

$$winner = argmax_X(min_Y score(X,Y))$$

- Decision: W > M > B

In case there is no Condorcet winner

- Coombs rule : similar to single transferable vote; eliminate the alternative which is ranked last by the greatest number of voters, until one remaining alternative gets over 50% of votes:

  7 : M > W > B

  9 : W > B > M

  4 : B > M > W

  ⇒

  Stage 1: eliminate M

  Stage 2: eliminate B, W is the winner

- Example: choice of the host of olympic games

Jean-Charles de Borda (1733-1799) – Borda rule

- Each DM gives n-1 points to the most preferred alternative, n-2 points to the second most preferred, ..., and 0 points to the least preferred alternative
- The alternative with the highest total number of points is the winner
- An example: 3 alternatives, 9 voters

4 states that $A > B > C$ $\quad$ $A : 4 \cdot 2 + 3 \cdot 0 + 2 \cdot 0 = 8$ votes

3 states that $B > C > A$ $\quad$ $B : 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 1 = 12$ votes

2 states that $C > B > A$ $\quad$ $C : 4 \cdot 0 + 3 \cdot 1 + 2 \cdot 2 = 7$ votes

⇨ B is the Borda winner

Generalization of Borda rule

- Positional scoring rule :
  - Vector of position scores: $s = \langle s_1, s_2, \ldots, s_n \rangle$, where $s_1 \geq s_2 \geq \ldots \geq s_n$
  - Borda rule: $\langle n-1, n-2, \ldots, 0 \rangle$
  - Plurality rule: $\langle 1, 0, \ldots, 0 \rangle$
  - Antiplurality rule: $\langle 1, \ldots, 1, 0 \rangle$
- Baldwin rule : in consecutive stages, eliminate the alternative with the worst Borda score:

7 : M > W > B

9 : W > B > M

4 : B > M > W

⇒

Stage 1: M=18, W=25, B=17, eliminate B

Stage 2: M=11, W=9, M is the winner

Ranking: M > W > B

Allocating seats in party-list proportional representation

- D'Hondt method (Poland, Austria, Finland, Israel, Spain, Netherlands) :
  - divide the number of obtained votes by natural numbers, n=1,2,3,...

party:	A	B	C
n=1	240	360	150
n=2	120	180	75
n=3	80	120	50
n=4	60	90	37.5
n=5	48	72	30

The number of seats to be shared

s=8

  - if **s** is the number of seats, order **s** results of the division according to decreasing values:

    360(B), 240(A), 180(B), 150(C), 120(B), 120(A), 90(B), 80(A)

  - assign to party X as many seats as the number of times X appears in the above order: B = 4 seats, A = 3 seats, C = 1 seat
  - in case of tie, take the party with the greatest number of votes, and then with the greatest number of winning electoral districts

Allocating seats in party-list proportional representation

- Sainte-Laguë method (Norway, Sweden, Danmark, Bosnia, Latvia, Kosowo, Germany, New Zealand, Poland in 2001) :
  - divide the number of obtained votes by odd numbers, n=1,3,5,...

party:	A	B	C
n=1	240	360	150
n=3	80	120	50
n=5	48	72	30
n=7	34.28	51.43	21.43

The number of seats to be shared

s=8

  - if **s** is the number of seats, order **s** results of the division according to decreasing values:

    360(B), 240(A), 150(C), 120(B), 80(A), 72(B), 51.43(B), 50(C)

  - assign to party X as many seats as the number of times X appears in the above order:

B = 4 seats, A = 2 seats, C = 2 seats

Coalitions

- If the voting procedure is known voters may form coalitions that serve their purposes
  - Eliminate an undesired alternative
  - Support a commonly agreed alternative

Weak preference order

- The opinion of the DM_i about two alternatives is called a weak preference order R_i:

  The DM_i thinks that x is at least as good as y $\Leftrightarrow$ x R_i y (outranking)

- How the collective preference R should be determined when there are k decision makers?
- What is the *social choice function* f that gives $R=f(R_1,\dots,R_k)$?
- Voting procedures are potential choices for social choice functions

Requirements on the social choice function (1/2)

1) Non trivial

There are at least two DMs and three alternatives

2) Complete and transitive R_i's

If $x \neq y \Rightarrow x R_i y \vee y R_i x$ (i.e. all DMs have an opinion)

If $x R_i y \wedge y R_i z \Rightarrow x R_i z$

3) *f* is defined for all R_i's

The group has a well defined preference relation, regardless of what the individual preferences are

Requirements on the social choice function (2/2)

4) Independence of irrelevant alternatives

The group's choice doesn't change if we add an alternative that is

- considered inferior to all other alternatives by all DMs, or
- is a copy of an existing alternative

5) Pareto principle

If all group members prefer x to y, the group should choose the alternative x

6) Non dictatorship

There is no DM_i such that $x \; R_i \; y \Rightarrow x \; R \; y$

Arrow's theorem

(Kenneth Arrow, 1921-)

Nobel Prize 1972

There is no complete and transitive social choice function *f* satisfying the conditions 1-6

Arrow's theorem - an example

Borda voting procedure:

	DM_1	DM_2	DM_3	DM_4	DM_5	total
x_1	3	3	1	2	1	10
$\mathbf{x_2}$	2	2	3	1	3	**11**
x_3	1	1	2	0	0	4
x_4	0	0	0	3	2	5

Alternative x_2 is the winner!

Suppose that DMs' preferences do not change.
A ballot between the alternatives 1 and 2 gives

	DM_1	DM_2	DM_3	DM_4	DM_5	total
$\mathbf{x_1}$	1	1	0	1	0	**3**
x_2	0	0	1	0	1	2

Alternative x_1 is the winner!

The fourth condition is not satisfied!

Value (utility) aggregation (1/2)

Theorem (Harsanyi [1994 Nobel Prize] 1955, Keeney 1975):

Let $v_i(\cdot)$ be a measurable marginal value function describing the preferences of DM_i. There exists a k-dimensional differentiable function $v_g()$ with positive partial derivatives describing group preferences $>_g$ in the definition space, such that

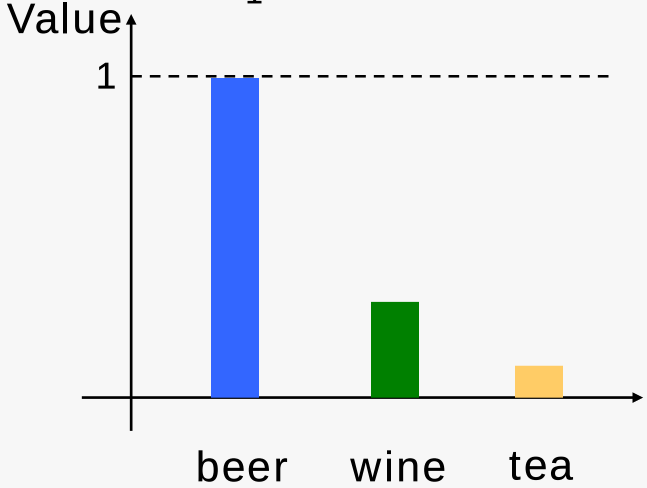
$$a >_g b \Leftrightarrow v_g[v_1(a),...,v_k(a)] \geq v_g[v_1(b),...,v_k(b)]$$

and conditions 1-6 are satisfied.

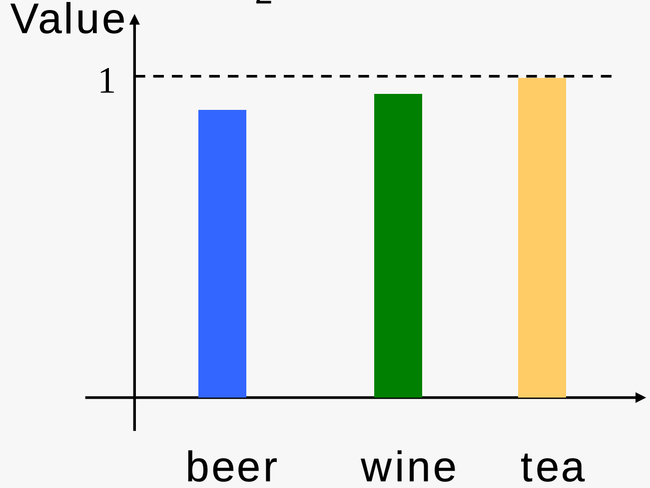
Value aggregation (2/2)

- In addition to the weak preference order also a cardinal scale describing the strength of the preferences is required

DM$_1$: beer > wine > tea



DM$_2$: tea > wine > beer



- Value function describes also the strength of the preferences

Problems in value aggregation

- There is a function describing group preferences but it may be difficult to define in practice
- Comparing the values of different DMs is not straightforward
- Solution:
  - Each DM defines her/his own value function
  - Group preferences are calculated as an aggregate (weighted sum?) of the individual preferences
- Unequal or equal weights?
  - Should the chairman get a higher weight
  - Group members can weight each others' expertise
  - Defining the weight is likely to be politically difficult (e.g. in EU)
  - Are the DMs preferentially independent?
- Use more complex aggregation models – loose in transparency?