Proper vs. Directed Interval Arithmetic in Solving the Poisson Equation

Tomasz Hoffmann, Andrzej Marciniak

Poznan University of Technology, Institute of Computing Science
Piotrowo 2, 60-965 Poznan, Poland

e-mail: tomaszhof@gmail.com, Andrzej.Marciniak@put.poznan.pl

Abstract

Many scientific and engineering problems are described in the form of partial differential equations. If such equations cannot be solved analytically, we use approximate methods to solve them, usually providing all calculations in floating-point arithmetic. Using approximate methods we obtain solutions including some errors of methods, and floating-point arithmetic causes representation errors and rounding errors. Interval arithmetic makes it possible to represent any input data in the form of machine interval and perform all calculations in floating-point interval arithmetic which includes rounding errors. If an interval method used to solve a problem includes also the error of the method, then we can obtain a solution (in the form of interval) which contains all possible numerical errors.

In our previous papers [2, 3] we have considered an interval difference method for solving the Poisson equation

$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=f(x,y). \tag{1}$$

In (1) the function f describes the input to the problem on a plane region R whose boundary will be denoted by Γ. We assume that this function is continuous together with its partial derivatives up to the second order.

To obtain a unique solution to (1), we usually apply the Dirichlet boundary conditions

$$u(x,y)=\varphi(x,y)$$

for all (x,y) on Γ. In general, the plane region R may be arbitrary, but further we will assume that R is a rectangular:

$$R=\{(x,y):\ 0<x<\alpha,\ 0<y<\beta\}.$$

Thus, the problem is to find $u = u(x, y)$ satisfying the equation

$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=f(x,y),\quad 0<x<\alpha,\quad 0<y<\beta, \tag{2}$$

with boundary conditions

$$u|_\Gamma=\varphi(x,y)=\begin{cases}\varphi_1(y) & \text{for } x=0,\\ \varphi_2(x) & \text{for } y=0,\\ \varphi_3(y) & \text{for } x=\alpha,\\ \varphi_4(x) & \text{for } y=\beta,\end{cases} \tag{3}$$

where

$$\varphi_1(0)=\varphi_2(0),\quad \varphi_2(\alpha)=\varphi_3(0),\quad \varphi_3(\beta)=\varphi_4(\alpha),\quad \varphi_4(0)=\varphi_1(\beta),$$
$$\Gamma=\{(x,y):\ x=0,\alpha \text{ and } 0\le y\le\beta \text{ or } 0\le x\le\alpha \text{ and } y=0,\beta\}.$$

Partitioning the interval $[0, \alpha]$ into n equal parts of width h and the interval $[0, \beta]$ into m equal parts of width k provides a means of placing a grid on the rectangle R with mesh points $(x_i, y_j) = (ih, jk)$, where $h = \alpha/n$, $k = \beta/m$, $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$. Assuming that the fourth order partial derivatives of u exist, for each mesh point in the interior of the grid we use the Taylor series in the variable x about x_i and in the variable y about y_j. This allows us to express the Poisson equation at the points (x_i, y_j) as

$$\begin{gathered}\frac{u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)}{h^2}+\frac{u(x_i,y_{j+1})-2u(x_i,y_j)+u(x_i,y_{j-1})}{k^2}\\ -\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,y_j)-\frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i,\eta_j)=f(x_i,y_j),\quad i=1,2,\ldots,n-1,\quad j=1,2,\ldots,m-1,\end{gathered}\tag{4}$$

where $\xi_i\in(x_{i-1},x_{i+1})$, $\eta_j\in(y_{j-1},y_{j+1})$, and the boundary conditions as

$$\begin{aligned}&u(0,y_j)=\varphi_1(y_j)\ \text{ for each } j=0,1,\ldots,m,\\ &u(x_i,0)=\varphi_2(x_i)\ \text{ for each } i=1,2,\ldots,n-1,\\ &u(\alpha,y_j)=\varphi_3(y_j)\ \text{ for each } j=0,1,\ldots,m,\\ &u(x_i,\beta)=\varphi_4(x_i)\ \text{ for each } i=1,2,\ldots,n-1.\end{aligned}\tag{5}$$

Omitting in (4) the partial derivatives, this results in a method, called the central-difference method, with local truncation error of order $O(h^2 + k^2)$.

Let us assume that there exists a constant M such that

$$\left|\frac{\partial^4 u}{\partial x^2\,\partial y^2}\right|\le M \ \text{ for all } 0\le x\le\alpha \text{ and } 0\le y\le\beta,\tag{6}$$

and let

$$\frac{\partial^4 u}{\partial x^2\,\partial y^2}(x,y)=\frac{\partial^4 u}{\partial y^2\,\partial x^2}(x,y).$$

Since from the Poisson equation (1) it follows that

$$\begin{aligned}\frac{\partial^4 u}{\partial x^4}(x,y)&=\frac{\partial^2 f}{\partial x^2}(x,y)-\frac{\partial^4 u}{\partial x^2\,\partial y^2}(x,y),\\ \frac{\partial^4 u}{\partial y^4}(x,y)&=\frac{\partial^2 f}{\partial y^2}(x,y)-\frac{\partial^4 u}{\partial y^2\,\partial x^2}(x,y),\end{aligned}$$

then it is obvious that we have

$$\begin{aligned}&\frac{\partial^4 u}{\partial x^4}(\xi,y)\in\Psi(X+[-h,h],Y)+[-M,M],\\ &\frac{\partial^4 u}{\partial y^4}(\xi,y)\in\Omega(X,Y+[-k,k])+[-M,M],\end{aligned}\tag{7}$$

where $\Psi(X, Y)$ and $\Omega(X, Y)$ denote interval extensions of $\frac{\partial^2 f}{\partial x^2}(x, y)$ and $\frac{\partial^2 f}{\partial y^2}(x, y)$, respectively. If we recall the Poisson equation at the mesh points (4) and write the partial derivatives at the right-hand side, it is easy now to write and interval equivalent to this equation. We have

$$k^2 U_{i-1, j} + h^2 U_{i, j-1} - 2(h^2 + k^2) U_{i, j} + k^2 U_{i+1, j} + h^2 U_{i, j+1}$$

$$= h^2 k^2 \left\{ F_{i, j} + \frac{1}{12} \left[h^2 \Psi(X_i + [-h, h], Y_j) + k^2 \Omega(X_i, Y_j + [-k, k]) + (h^2 + k^2)[-M, M] \right] \right\}, \quad (8)$$

$$i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m-1,$$

where $F_{i,j} = F(X_i, Y_j)$, and where

$$\begin{aligned} U_{0, j} &= \Phi_1(Y_j) \text{ for each } j = 0, 1, \dots, m, \\ U_{i, 0} &= \Phi_2(X_i) \text{ for each } i = 1, 2, \dots, n-1, \\ U_{n, j} &= \Phi_3(Y_j) \text{ for each } j = 0, 1, \dots, m, \\ U_{i, m} &= \Phi_4(X_i) \text{ for each } i = 1, 2, \dots, n-1. \end{aligned} \quad (9)$$

($\Phi_1(Y)$, $\Phi_2(X)$, $\Phi_3(Y)$ and $\Phi_4(X)$ denote interval extensions of the functions $\varphi_1(y)$, $\varphi_2(x)$, $\varphi_3(y)$ and $\varphi_4(x)$, respectively.)

The system of linear interval equations (8) – (9) can be solved in conventional (proper) floating-point interval arithmetic (see e.g. [1]) since all intervals are proper, i.e. for any interval $[a, b]$ we have $a \le b$.

But we can consider another interval equivalent of (4). Namely, we can write (also using (7))

$$k^2 U_{i-1, j} + h^2 U_{i, j-1} - 2(h^2 + k^2) U_{i, j} + k^2 U_{i+1, j} + h^2 U_{i, j+1}$$

$$- \frac{h^2 k^2}{12} \left\{ h^2 \Psi(X_i + [-h, h], Y_j) + k^2 \Omega(X_i, Y_j + [-k, k]) + (h^2 + k^2)[-M, M] \right\}$$

$$= h^2 k^2 F_{i, j}, \quad i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m-1.$$

Using directed interval arithmetic (see e.g. [4] and [5]), we can add at both sides of this equation the opposites to

$$-\frac{h^4 k^2}{12} \Psi(X_i + [-h, h], Y_j), \ -\frac{h^2 k^4}{12} \Omega(X_i, Y_j + [-k, k]) \text{ and } -\frac{h^2 k^2}{12}(h^2 + k^2)[-M, M]$$

(the opposite of an interval, like the inverse of an interval, does not exist in proper interval arithmetic). We get

$$k^2 U_{i-1, j} + h^2 U_{i, j-1} - 2(h^2 + k^2) U_{i, j} + k^2 U_{i+1, j} + h^2 U_{i, j+1}$$

$$= h^2 k^2 \left\{ F_{i, j} + \frac{1}{12} \left[h^2 \Psi(X_i + [-h, h], Y_j) + k^2 \Omega(X_i, Y_j + [-k, k]) + (h^2 + k^2)[M, -M] \right] \right\}, \quad (10)$$

$$i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m-1.$$

The equation (10) differs from the equation (8) only by the last term on the right-hand side which is an improper interval. But using the directed floating-point interval arithmetic we can solve the system of equations (10) (together with (9)). If the interval solutions of this system are in the form of improper intervals, to get the proper intervals we can use the so-called proper projection of intervals, i.e. transform each interval $[a, b]$, for which $b < a$, to the interval $[b, a]$.

We have carried out a number of numerical experiments for various functions $f(x, y)$ occurring in the Poisson equation (2) and various boundary conditions (3) using both: the method (8) with the conventional floating-point interval arithmetic and the method (10) with the directed floating-point interval arithmetic. In all examples considered and in both these methods, the exact solutions (if they are known) are included in the interval solutions obtained. It is important that although the calculations by the method (10) in directed floating-point interval arithmetic are longer in time than by the method (8) in conventional one, the method (10) yields interval solutions with smaller widths.

References

[1] Hammer, R., Hocks, M., Kulisch, U., Ratz, D., *Numerical Toolbox for Verified Computing I: Basic Numerical Problems*, Springer, Berlin 1993.

[2] Hoffmann, T., Marciniak, A., Solving the Poisson Equation by an Interval Difference Method of the Second Order, *Computational Methods in Science and Technology* (to appear).

[3] Marciniak, A., An Interval Difference Method for Solving the Poisson Equation – the First Approach, *Pro Dialog* 24 (2008), 49–61.

[4] Markov, S., On Directed Interval Arithmetic and its Applications, *Journal of Universal Computer Science* 7 (1995), 514–526.

[5] Popova, E. D., Extended Interval Arithmetic in IEEE Floating-Point Environment, *Interval Computations* 4 (1994), 100–129.