INTERVAL DIFFERENCE METHODS FOR SOLVING THE POISSON EQUATION

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INTRODUCTION

As it is well-known, there are two kinds of errors caused by floating-point arithmetic:

- ★representation errors,
- ★rounding errors.

When we apply an approximate method to solve a problem on a computer we introduce the third kind of error:

- ★the error of method (usually called the truncation error.

Using interval methods realized in interval floating--point arithmetic we can obtain solutions (in the form of intervals) which cantain all these errors.

INTRODUCTION

This presentation is devoted to interval difference methods (of second and fourth order) for solving the Poisson equation with boundary conditions.

The solutions are in the form of intervals which contain all possible numerical errors. Moreover, it has been experimentally confirmed*) that the exact solutions are placed inside the resulting intervals.

Numerical examples have been carried out in proper and directed interval arithmetics using our *IntervalArithmetic32and64* unit written in the Delphi Pascal programming language.

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*) In our opinion, it is rather impossible to obtain a theoretical proof of this fact.

INTERVAL ARITHMETIC

Verified numerical computing requires a mathematical tool to describe operations performed on computers. Such a mathematical tool, called interval arithmetic, has been developed by R. E. Moore in 1966 and extended by other researchers in the following years.

As it is well-known, a *real interval*, or shortly an *interval*, is a closed and bounded subset of real numbers $\mathbf{R}$:

$$[x] = [\underline{x}, \overline{x}] = \{x \in \mathbf{R} : \underline{x} \leq x \leq \overline{x}\},$$

where $\underline{x}$ and $\overline{x}$ denote the lower and upper bounds of the interval $[x]$, respectively. An interval is called a *point interval* if $\underline{x} = \overline{x}$. The set of real intervals we will denote by $\mathbf{IR}$.

INTERVAL ARITHMETIC

The elementary real operations (addition, subtraction, multiplication and division), i.e. any operation $\circ \in \{+, -, \cdot, /\}$, can be extended to interval arguments $[x]$, $[y]$ by defining the result of an elementary interval operation to be the set of real numbers which results from combining any two numbers included in intervals $[x]$ and $[y]$:

$$[x] \circ [y] = \{ x \circ y : \ x \in [x], y \in [y] \}. \qquad (1)$$

From (1) it follows that

$$\begin{aligned}
[x] + [y] &= \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right], \\
[x] - [y] &= \left[\underline{x} - \overline{y}, \overline{x} - \underline{y}\right], \\
[x] \cdot [y] &= \left[\min\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}, \max\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}\right], \\
[x] / [y] &= [x] \cdot \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right], \quad 0 \notin [y].
\end{aligned} \qquad (2)$$

INTERVAL ARITHMETIC

The real interval arithmetic can be extended to complex intervals. We omit adequate definitions, becauce futher only real interval operations will be applied.

In real interval arithmetic it is excluded a division by an interval containing zero. This restriction may be removed in so called *extended* (*real*) *interval arithmetic* which is defined in the set of extended real intervals

$$\mathbf{IR}^* = \mathbf{IR} \cup \{[-\infty, r] :\ r \in \mathbf{R}\} \cup \{[l, +\infty] :\ l \in \mathbf{R}\} \cup \{[-\infty, +\infty]\}.$$

Both of the above interval atithmetic (real and extended real) are called *proper*, since for any interval $[x] = [\underline{x}, \bar{x}]$ we have $\underline{x} \le \bar{x}.$

It should be noted that the opposite and the inverse elements do not exist in proper interval arithmetic.

Such elements exist in so called *directed interval arithmetic*, where for any interval $[x]$ we can have either $\underline{x} \le \bar{x}$ or $\underline{x} \ge \bar{x}.$

INTERVAL ARITHMETIC

For any real-valued function $f : D \subset \mathbf{R} \to \mathbf{R}$ we can extend it to interval arguments $[x] \in D$:

$$f([x]) = \bigcup_{x \in [x]} f(x).$$

An enclosure of $f([\mathrm{x}])$ can be easy obtained if we substitute $[x]$ for x in the defining expression of f, and then evaluate f using interval arithmetic. This kind of evaluation is called an *interval extension* of f and is denoted by $f_{[\,]}([x])$. It should be noted that in general we have

$$f([x]) \subseteq f_{[\,]}([x]).$$

Moreover, one should know that a real-valued function may have several interval extensions, since it may be defined by several equivalent arithmetic expressions and such expressions do not necessarily yield equivalent interval extensions. In general, it is difficult to determine the best possible interval extension if we have a few mathematical equivalent notations of a real-valued function. However, it is empirical fact that the fewer occurrences of $[x]$ within an interval extension, the better is the result of the corresponding interval evaluation.

INTERVAL ARITHMETIC

The realization of proper interval arithmetic is based on simple rule, where left and right endpoints are calculated by using downward and upward rounding, respectively, i.e. (compare (2))

$$[x]+[y]=\left[\nabla\left(\underline{x}+\underline{y}\right),\Delta\left(\overline{x}+\overline{y}\right)\right],$$

$$[x]-[y]=\left[\nabla\left(\underline{x}-\overline{y}\right),\Delta\left(\overline{x}-\underline{y}\right)\right],$$

$$[x]\cdot[y]=\Big[\min\left\{\nabla\left(\underline{x}\underline{y}\right),\nabla\left(\underline{x}\overline{y}\right),\nabla\left(\overline{x}\underline{y}\right),\nabla\left(\overline{x}\overline{y}\right)\right\},$$
$$\max\left\{\Delta\left(\underline{x}\underline{y}\right),\Delta\left(\underline{x}\overline{y}\right),\Delta\left(\overline{x}\underline{y}\right),\Delta\left(\overline{x}\overline{y}\right)\right\}\Big],$$

$$[x]/[y]=\Big[\min\left\{\nabla\left(\underline{x}/\underline{y}\right),\nabla\left(\underline{x}/\overline{y}\right),\nabla\left(\overline{x}/\underline{y}\right),\nabla\left(\overline{x}/\overline{y}\right)\right\},$$
$$\max\left\{\Delta\left(\underline{x}/\underline{y}\right),\Delta\left(\underline{x}/\overline{y}\right),\Delta\left(\overline{x}/\underline{y}\right),\Delta\left(\overline{x}/\overline{y}\right)\right\}\Big],\quad 0\notin[y],$$

INTERVAL ARITHMETIC

In the case of directed interval arithmetic the rules of calculating endpoints are much more complicated. For each basic operation different rounding can be used for calculation of endpoints of the result interval.

The accurate description of directed interval arithmetic is presented, among others, in:

- ★ E. D. Popowa, Extended Interval Arithmetic in IEEE Floating-Point Environment, *Interval Computations* 4 (1994), 100-129,
- ★ A. Marciniak, On Realization of Floating-Point Directed Interval Arithmetic, http://www.cs.put.poznan.pl/amarciniak/KONF-referaty/DirectedArithmetic.pdf, 2012.

THE POISSON EQUATION

An elliptical partial-differential equation, known as the Poisson equation, is of the form

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y). \qquad (3)$$

We assume that the function f describes the input to the problem on a plane region R whose boundary will be denoted by Γ.

Equations of this type arise naturally in the study of various time-independent problems such as:

- ⋆ two-dimensional steady-state problems involving incompressible fluids,
- ⋆ the potential energy of a plane acted by gravitational forces in the plane,
- ⋆ the steady-state distribution of heat in a plane region.

THE POISSON EQUATION

To obtain a unique solution to the Poisson equation, additional constraints must be placed to the solution. Usually, we apply the Dirichlet boundary conditions, given by

$$u(x, y) = \varphi(x, y)$$

for all (x, y) on Γ. In general, the plane region R may be arbitrary. Further, we will assume that R is a rectangular:

$$R = \{(x, y): 0 \le x \le \alpha, 0 \le y \le \beta\}.$$

THE POISSON EQUATION

Thus, our problem is to find $u = u(x, y)$ satisfying the partial-
-differential equation

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y), \quad 0 \le x \le \alpha, \quad 0 \le y \le \beta,$$

with the boundary conditions

$$u\big|_{\Gamma} = \varphi(x, y) = \begin{cases} \varphi_1(y) & \text{for } x = 0, \\ \varphi_2(x) & \text{for } y = 0, \\ \varphi_3(y) & \text{for } x = \alpha, \\ \varphi_4(x) & \text{for } y = \beta, \end{cases}$$

where

$$\varphi_1(0) = \varphi_2(0), \quad \varphi_2(\alpha) = \varphi_3(0), \quad \varphi_3(\beta) = \varphi_4(\alpha), \quad \varphi_4(0) = \varphi_1(\beta),$$
$$\Gamma = \{(x, y) \colon x = 0, \alpha \text{ and } 0 \le y \le \beta \text{ or } 0 \le x \le \beta \text{ and } y = 0, \beta\}.$$

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

Partitioning the interval $[0, \alpha]$ into n equal parts of width h and the interval $[0, \beta]$ into *m* equal parts of width *k* provides a mean of placing a grid on the rectangle R with mesh points $(x_i, y_i) = (ih, jk)$, where $h = \alpha/n$, $k = \beta/m$, $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$. Assuming that the fourth order partial derivatives of u exists, for each mesh point in the interior of the grid we use the Taylor series in the variable x about x_i and in the variable y about y_j. This allows us to express the Poisson equation at the points (x_i, y_j) as

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) - \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j) = f(x_i, y_j), \quad (4)$$

$$i = 1, 2, \ldots, n-1, \quad j = 1, 2, \ldots, m-1,$$

where

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

$$\delta_x^2 u_{ij} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}, \quad \delta_y^2 u_{ij} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2},$$

$u_{ij} = u(x_i, y_j)$, and where $\xi_i \in (x_{i-1}, x_{i+1})$, $\eta_j \in (y_{j-1}, y_{j+1})$ are intermidiate points, and the boundary conditions as

$$\begin{aligned} u(0, y_j) &= \varphi_1(y_j) \text{ for each } j = 0, 1, \dots, m, \\ u(x_i, 0) &= \varphi_2(x_i) \text{ for each } i = 1, 2, \dots, n-1, \\ u(\alpha, y_j) &= \varphi_3(y_j) \text{ for each } j = 0, 1, \dots, m, \\ u(x_i, \beta) &= \varphi_4(x_i) \text{ for each } i = 1, 2, \dots, n-1. \end{aligned} \tag{5}$$

Omitting in (4) the partial derivatives, we obtain a method, called *the central-difference method*, with local truncation error of order $O(h^2 + k^2)$. Such formulas together with (5) present a system of linear equations (with respect to unknowns u_{ij}) which may be solved by any known exact or iterative method.

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

To construct an interval method, let us assume that there exists a constant M such that

$$\left|\frac{\partial^4 u}{\partial x^2 \partial y^2}\right| \le M \text{ for all } 0 \le x \le \alpha \text{ and } 0 \le y \le \beta,$$

and let

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x, y) = \frac{\partial^4 u}{\partial y^2 \partial x^2}(x, y).$$

Since from the Poisson equation (3) it follows that

$$\frac{\partial^4 u}{\partial x^4}(x, y) = \frac{\partial^2 f}{\partial x^2}(x, y) - \frac{\partial^4 u}{\partial x^2 \partial y^2}(x, y),$$

$$\frac{\partial^4 u}{\partial y^4}(x, y) = \frac{\partial^2 f}{\partial y^2}(x, y) - \frac{\partial^4 u}{\partial y^2 \partial x^2}(x, y),$$

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

then it is obvious that we have

$$\frac{\partial^4 u}{\partial x^4}(\xi, y) \in \Psi(X + [-h, h], Y) + [-M, M],$$

$$\frac{\partial^4 u}{\partial y^4}(x, \eta) \in \Omega(X, Y + [-k, k]) + [-M, M],$$

for any $\xi \in (x - h, x + h)$ and any $\eta \in (y - k, y + k)$, where X and Y denote interval extension of x and y, respectively, and $\Psi(X, Y)$ and $\Omega(X, Y)$ are interval extension of $\frac{\partial^2 f}{\partial x^2}(x, y)$ and $\frac{\partial^2 f}{\partial y^2}(x, y)$, respectively.

If we recall the Poisson equation at the mesh points (4) and write the partial derivatives at the right-hand side, it is easy to write an interval analogy to this equation.

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

Assuming that all interval extensions are proper, we have

$$k^2U_{i-1,j} + h^2U_{i,j-1} - 2(h^2 + k^2)U_{i,j} + k^2U_{i+1,j} + h^2U_{i,j+1}$$
$$= h^2k^2\left(F_{i,j} + \frac{1}{12}\left(h^2\Psi(X_i + [-h,h], Y_j) + k^2\Omega(X_i, Y_j + [-k,k])\right.\right.$$
$$\left.\left. + (h^2 + k^2)[-M, M]\right)\right), \quad (6)$$
$$i = 1, 2, \ldots, n-1, \quad j = 1, 2, \ldots, m-1,$$

where $F_{i,j} = F(X_i, Y_j)$, and where

$$U_{0,j} = \Phi_1(Y_j), \quad U_{i,0} = \Phi_2(X_i), \quad U_{n,j} = \Phi_3(Y_j), \quad U_{i,m} = \Phi_4(X_i)$$
$$\text{for each } j = 0, 1, \ldots, m \text{ and } i = 1, 2, \ldots, n-1, \quad (7)$$

$\Phi_1(Y)$, $\Phi_2(X)$, $\Phi_3(Y)$ and $\Phi_4(X)$ denote interval extensions of the function $\varphi_1(y)$, $\varphi_2(x)$, $\varphi_3(y)$ and $\varphi_4(x)$, respectively.

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

The system of linear equations (6) – (7) can be solved in conventional (proper) floating-point interval arithmetic, because all intervals are proper.

But we can consider another analogy of (4). Namely, we can write

$$k^2 U_{i-1,j} + h^2 U_{i,j-1} - 2(h^2 + k^2) U_{i,j} + k^2 U_{i+1,j} + h^2 U_{i,j+1}$$

$$-\frac{h^2 k^2}{12}\Big(h^2 \Psi(X_i + [-h, h], Y_j) + k^2 \Omega(X_i, Y_j + [-k, k])$$

$$+ (h^2 + k^2)[-M, M]\Big) = h^2 k^2 F_{i,j},$$

$$i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m-1.$$

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

Using directed interval arithmetic, we can add at both sides of this equation the opposites to

$$-\frac{h^4k^2}{12}\Psi(X_i+[-h,h],Y_j),\quad -\frac{h^2k^4}{12}\Omega(X_i,Y_j+[-k,k])$$

and

$$-\frac{h^2k^2}{12}(h^2+k^2)[-M,M].$$

We get

$$k^2U_{i-1,j}+h^2U_{i,j-1}-2(h^2+k^2)U_{i,j}+k^2U_{i+1,j}+h^2U_{i,j+1}$$
$$=h^2k^2\Big(F_{i,j}+\frac{1}{12}\big(h^2\Psi(X_i+[-h,h],Y_j)+k^2\Omega(X_i,Y_j+[-k,k])$$
$$+(h^2+k^2)[M,-M]\big)\Big), \tag{8}$$
$$i=1,2,\ldots,n-1,\quad j=1,2,\ldots,m-1.$$

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

The last equation (8) differs from the equation (6) only by the last term on the right-hand side which is an improper interval. But using the directed interval floating-point arithmetic we can solve the system (8) (together with (7)). If the interval solutions of this system are in the form of improper intervals, to get the proper intervals we can use the so-called proper projecton of intervals, i.e. transform each interval $[a, b]$, for which $b < a$, to the interval $[b, a]$.

We should also add a remark concerning the constant M. In general, when the exact solution is unknown and nothing can be concluded about M from physical or technical properties or characteristics of the problem considered, we propose to find this constant by the following procedure:

INTERVAL DIFFERENCE METHODS OF SECOND ORDER

It is obvious that

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x_i, y_j) = \lim_{h\to 0}\lim_{k\to 0}\left(\frac{u_{i-1,\,j-1} + u_{i-1,\,j+1} + u_{i+1,\,j-1} + u_{i+1,\,j+1}}{h^2k^2} + \frac{4u_{i,\,j} - 2(u_{i-1,\,j} + u_{i,\,j-1} + u_{i,\,j+1} + u_{i+1,\,j}}{h^2k^2}\right).$$

We can calculate

$$M_{n,m} = \frac{1}{h^2k^2}\max_{i,\,j}\left|u_{i-1,\,j-1} + u_{i-1,\,j+1} + u_{i+1,\,j-1} + u_{i+1,\,j+1} + 4u_{i,\,j} - 2(u_{i-1,\,j} + u_{i,\,j-1} + u_{i,\,j+1} + u_{i+1,\,j})\right|$$

for i = 1, 2, ... , n – 1, j = 1, 2, ... , m – 1 and where u_{ij} are obtained by a conventional method for a variety of n and m, say n = m = 10, 20, ... , N, where N is sufficiently large. Then, we can plot $M_{n,\,m}$ against different n = m. The constant M can be easy determine from the obtained graph, since $\lim_{n\to\infty}\lim_{m\to\infty} M_{n,m} \le M.$

INTERVAL DIFFERENCE METHODS OF FOURTH ORDER

Using the Taylor series of higher order, we can express the Poisson equation at the points (x_i, y_j) as

$$
\begin{aligned}
&\delta_x^2 u_{ij} + \delta_y^2 u_{ij} + \frac{1}{12}(h^2 + k^2)\delta_x^2 \delta_y^2 u_{ij} \\
&\quad - \frac{1}{240}\left(h^4 \frac{\partial^6 u}{\partial x^4 \partial y^2}(\xi_i, y_j) + k^4 \frac{\partial^6 u}{\partial x^2 \partial y^4}(x_i, \eta_j) \right) \\
&\quad - \frac{h^2 k^2}{144}\left(\frac{\partial^6 u}{\partial x^4 \partial y^2}(\xi_i, \eta_j) + \frac{\partial^6 u}{\partial x^2 \partial y^4}(\xi_i, \eta_j) \right) \\
&= f_{ij} + \frac{1}{12}(h^2 \delta_x^2 + k^2 \delta_y^2) f_{ij} \\
&\quad - \frac{1}{240}\left(h^4 \frac{\partial^4 f}{\partial x^4}(\xi_i, y_j) + k^4 \frac{\partial^4 f}{\partial y^4}(x_i, \eta_j) \right).
\end{aligned}
\tag{9}
$$

INTERVAL DIFFERENCE METHODS OF FOURTH ORDER

Let $\Theta(X, Y)$ and $\Xi(X, Y)$ denote interval extensions of $\frac{\partial^4 f}{\partial x^4}(x, y)$ and $\frac{\partial^4 f}{\partial y^4}(x, y)$, respectively, and let us assume that

$$\left|\frac{\partial^6 u}{\partial x^4 \partial y^2}\right| \le P \text{ and } \left|\frac{\partial^6 u}{\partial x^2 \partial y^4}\right| \le Q \text{ for all } 0 \le x \le \alpha \text{ and } 0 \le y \le \beta.$$

It is obvious that

$$\frac{\partial^4 f}{\partial x^4}(\xi, y) \in \Theta(X + [-h, h], Y),$$

$$\frac{\partial^4 f}{\partial y^4}(x, \eta) \in \Xi(X, Y + [-k, k]),$$

$$\frac{\partial^6 u}{\partial x^4 \, \partial y^2} \in [-P, P], \quad \frac{\partial^6 u}{\partial x^2 \, \partial y^4} \in [-Q, Q].$$

INTERVAL DIFFERENCE METHODS OF FOURTH ORDER

If in (9) we write all partial derivatives at the right-hand side, then it is easy to obtain an interval analogy to this equation. We have

$$
\begin{aligned}
&(h^2+k^2)(U_{i-1,\,j-1}+U_{i-1,\,j+1}+U_{i+1,\,j-1}+U_{i+1,\,j+1})\\
&\quad +2(5k^2-h^2)(U_{i-1,\,j}+U_{i+1,\,j})+2(5h^2-k^2)(U_{i,\,j-1}+U_{i,\,j+1})\\
&\quad -20(h^2+k^2)U_{i,\,j}\\
&=h^2k^2\Big(F_{i-1,\,j}+F_{i+1,\,j}+8F_{i,\,j}+F_{i,\,j-1}+F_{i,\,j+1}\\
&\qquad\qquad -\frac{1}{20}\Big(h^4\Theta(X_i+[-h,h],Y_j)+k^4\Xi(X_i,Y_j+[-k,k])\Big)\\
&\qquad\qquad +\frac{1}{20}(h^4[-P,P]+k^4[-Q,Q])+\frac{h^2k^2}{12}[-P-Q,P+Q]\Big).
\end{aligned}
$$

INTERVAL DIFFERENCE METHODS OF FOURTH ORDER

If in (9) we leave partial derivatives at the left-hand side, write an interval analogy to this equation, and then add adequate opposite interval elements (which exist in directed interval aritmetic), we get

$$
\begin{aligned}
&(h^2+k^2)(U_{i-1,\,j-1}+U_{i-1,\,j+1}+U_{i+1,\,j-1}+U_{i+1,\,j+1})\\
&\quad +2(5k^2-h^2)(U_{i-1,\,j}+U_{i+1,\,j})+2(5h^2-k^2)(U_{i,\,j-1}+U_{i,\,j+1})\\
&\quad -20(h^2+k^2)U_{i,\,j}\\
&=h^2k^2\Big(F_{i-1,\,j}+F_{i+1,\,j}+8F_{i,\,j}+F_{i,\,j-1}+F_{i,\,j+1}\\
&\qquad -\frac{1}{20}\Big(h^4\Theta(X_i+[-h,h],Y_j)+k^4\Xi(X_i,Y_j+[-k,k])\Big)\\
&\qquad +\frac{1}{20}(h^4[P,-P]+k^4[Q,-Q])+\frac{h^2k^2}{12}[P+Q,-P-Q]\Big).
\end{aligned}
$$

The difference occurs only in the last line.

NUMERICAL EXAMPLES

Example 1

Let us take into account the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$

$$u\big|_{\Gamma} = \varphi(x, y) = \begin{cases} \varphi_1(y) = \cos(3y) \ \text{ for } \ x = 0, \\ \varphi_2(x) = \exp(3x) \ \text{ for } \ y = 0, \\ \varphi_3(y) = \exp(3)\cos(3y) \ \text{ for } \ x = 1, \\ \varphi_4(x) = \exp(3x)\cos(3) \ \text{ for } \ y = 1. \end{cases} \tag{10}$$

The exact solution is given by

$$u(x, y) = \exp(3x)\cos(3y)$$

and is presented in Figure 1.

NUMERICAL EXAMPLES

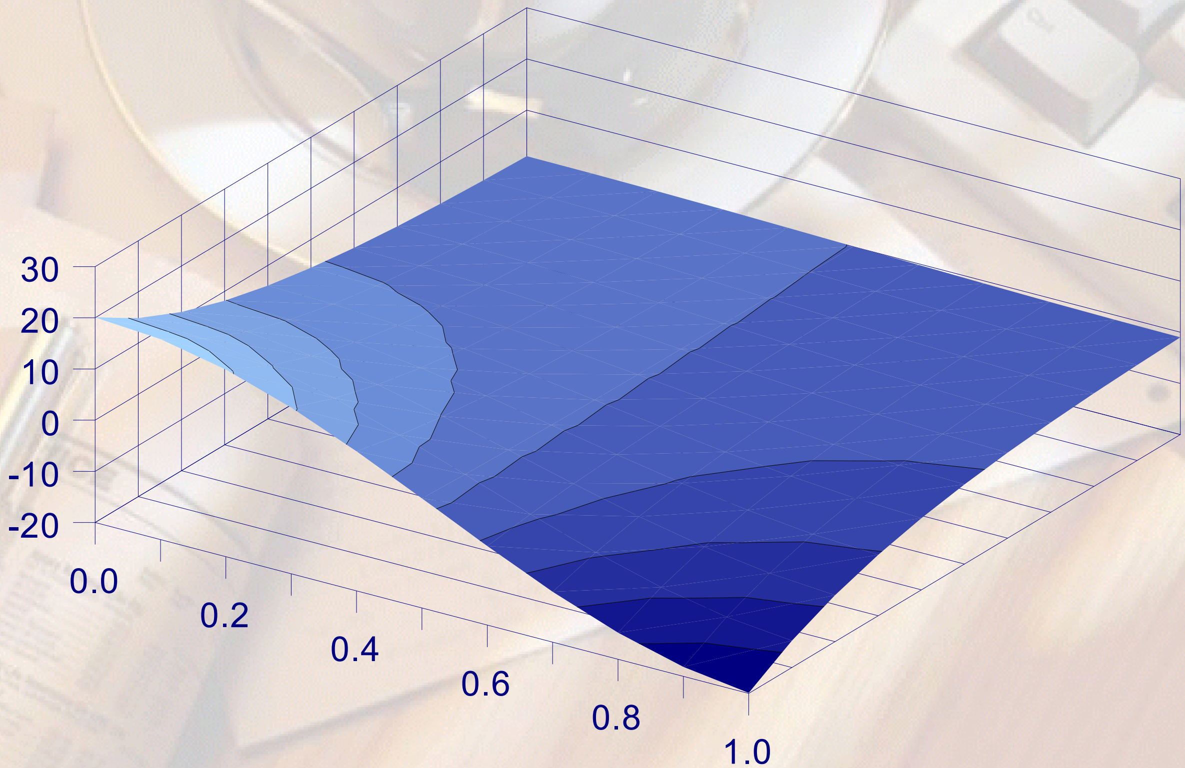

Figure 1. The solution of the problem (10)

In Table 1 we present the results obtained by the second and fourth order methods in proper and directed arithmetic at the center of the region Γ. The widths of intervals for the second order method are also presented in Figure 2.

NUMERICAL EXAMPLES

Table 1. The interval solutions and the widths of intervals obtained in proper (U_p) and directed (U_d) interval arithmetic to the problem (10) at $(0.5, 0.5)$ ($u_{exact}(0.5, 0.5) \approx 0.3170221435804436$6)

$n = m$	$U_p\,(0.5, 0.5)$	width(U_p)	$U_d\,(0.5, 0.5)$	width(U_d)
20 (2^{nd} order)	[0.26795781801796551, 0.36764778128690462]	0.099689963	[0.26795781801796628, 0.36764778128690385]	0.099689963
20 (4^{th} order)	[0.31687231501883790, 0.31717197371330709]	0.000299659	[0.31687231501883870, 0.31717197371330630]	0.000299659
60 (2^{nd} order)	[0.31156101681974897, 0.32265704145441798]	0.011096024	[0.31156101681975913, 0.32265704145440782]	0.011096024
60 (4^{th} order)	[0.31702029383932179, 0.31702399332372090]	0.000003700	[0.31702029383933559, 0.31702399332370710]	0.000003699
100 (2^{nd} order)	[0.31505586246198510, 0.31905099073825598]	0.003995128	[0.31505586246202793, 0.31905099073821316]	0.003995128
100 (4^{th} order)	[0.31702190385388648, 0.31702238330710142]	0.000000048	[0.31702212784104150, 0.31702215931994640]	0.000000031

NUMERICAL EXAMPLES

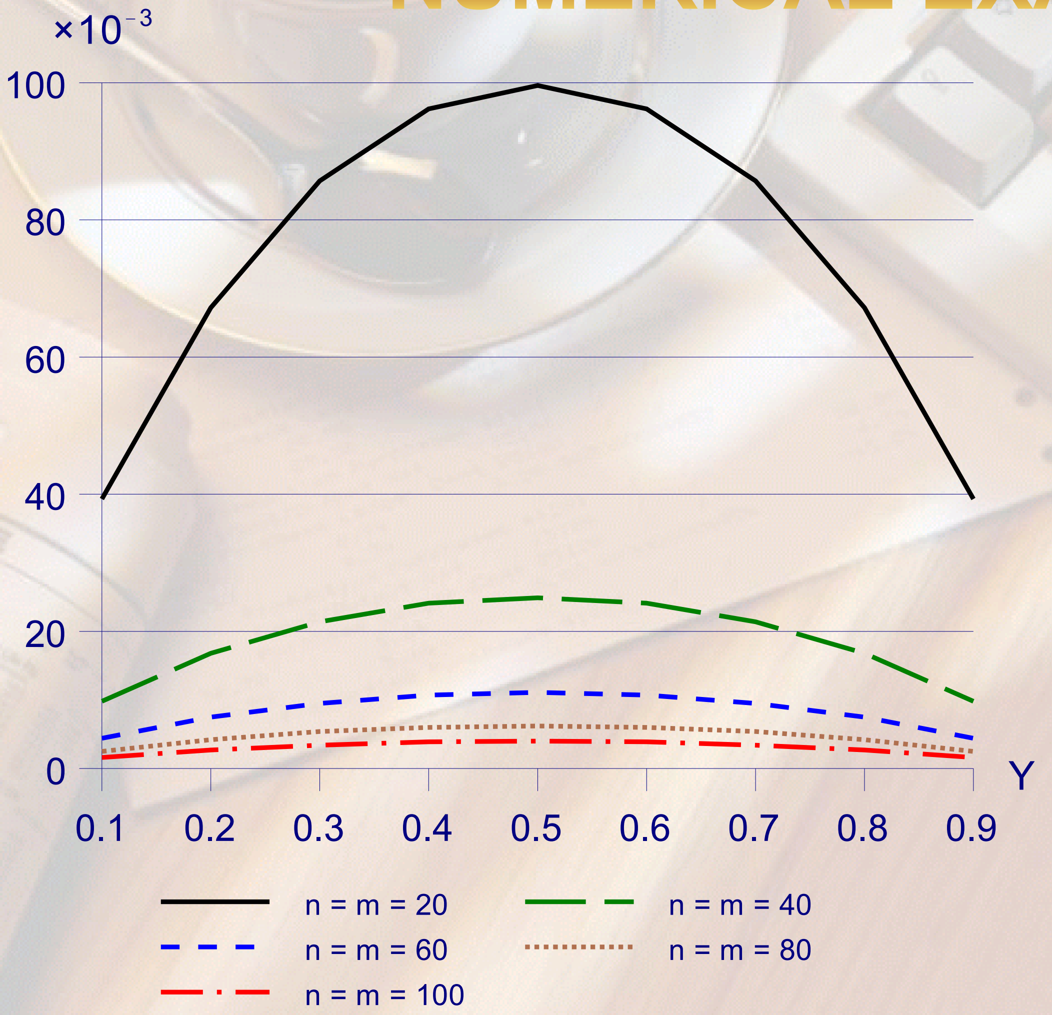

Figure 2. The widths of interval solutions to the problem (10) for the second order method on the grid $X = 0.5$ (for the fourth order method the shapes of curves are similar)

NUMERICAL EXAMPLES

In the second order methods we have assumed $M = 1627$. Of course, this estimation of $\left|\partial^4 u / (\partial x^2 \partial y^2)\right|$ can be calculated from the known exact solution, but a similar estimation one can obtain from the graph presented in Figure 3.

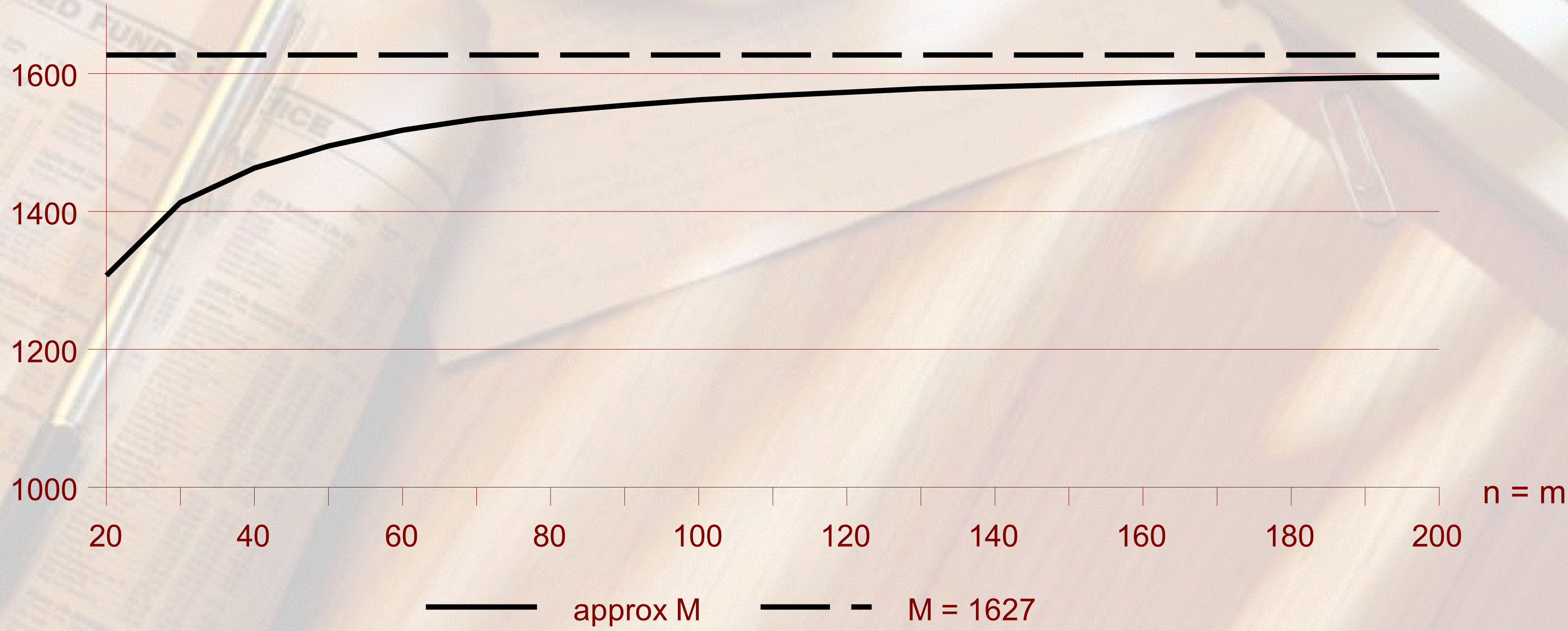

Figure 3. Approximations to the constant M for the problem (10)

NUMERICAL EXAMPLES

In the fourth order methods we have taken $P = Q = 14643$.
In general, if the estimations of $\left|\partial^6 u / (\partial x^4 \partial y^2)\right|$ and $\left|\partial^6 u / (\partial x^2 \partial y^4)\right|$ can not be obtained from any information about the problem considered, we can use similar technique as previously.

Example 2

As the second example let us consider the following problem:

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = -2\pi \sin(\pi x)\sin(\pi y), \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u\big|_{\Gamma} = 0, \tag{11}$$

with the exact solution (see Figure 4)

$$u(x, y) = \sin(\pi x)\sin(\pi y).$$

NUMERICAL EXAMPLES

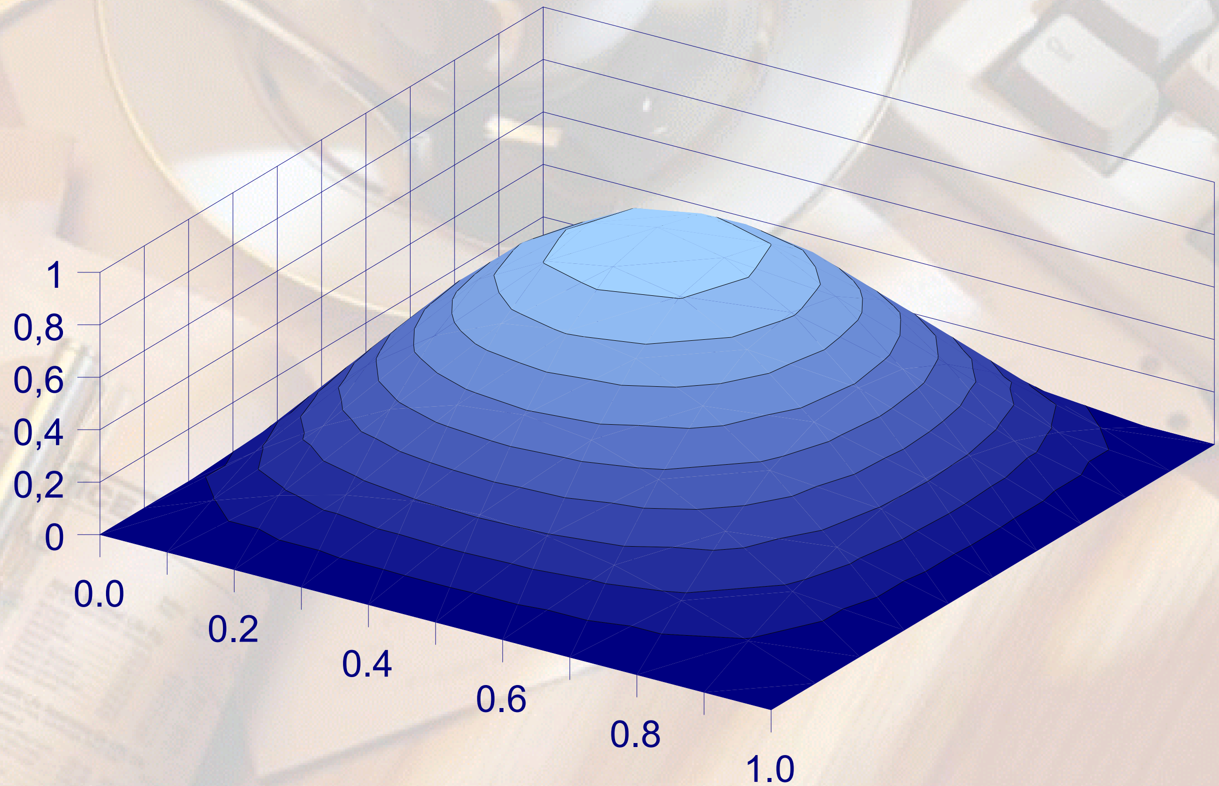

Figure 4. The solution of the problem (11)

The interval solutions obtained are presented in Table 2. The widths of intervals are also presented in Figure 5.

NUMERICAL EXAMPLES

Table 2. The interval solutions and the widths of intervals obtained in proper (U_p) and directed (U_d) interval arithmetic to the problem (11) at $(0.5, 0.5)$ ($u_{exact}(0.5, 0.5) = 1$)

$n = m$	$U_p\,(0.5, 0.5)$	width(U_p)	$U_d\,(0.5, 0.5)$	width(U_d)
20 (2^{nd} order)	[0.9943031722943299, 1.0032966998827956]	0.008993528	[0.9972920186287353, 1.0003078535483902]	0.003015835
20 (4^{th} order)	[0.9999825795708284, 1.0000059858600965]	0.000023406	[0.9999863114853876, 1.0000022539455373]	0.000015942
60 (2^{nd} order)	[0.9993877757476903, 1.0001910675167757]	0.000803292	[0.9995227144656405, 1.0000561287988255]	0.000533414
60 (4^{th} order)	[0.9999997879937084, 1.0000000498551452]	0.000000262	[0.9999998069620024, 1.0000000308868512]	0.000000217
100 (2^{nd} order)	[0.9997852097215218, 1.0000562138028589]	0.000259718	[0.9998155730093303, 1.0000258505150504]	0.000210278
100 (4^{th} order)	[0.9999999728030543, 1.0000000058410797]	0.000000033	[0.9999999743621285, 1.0000000042820054]	0.000000030

NUMERICAL EXAMPLES

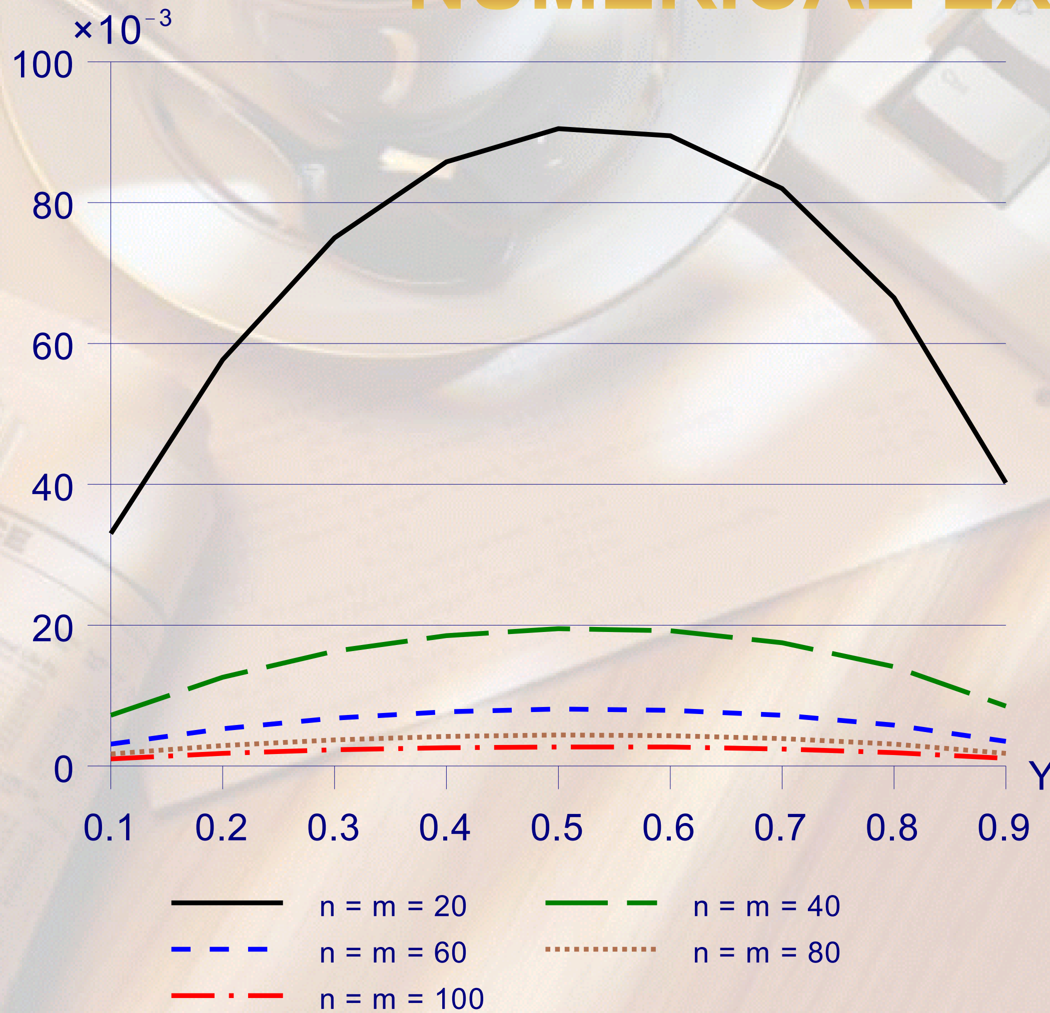

Figure 5. The widths of interval solutions to the problem (11) for the second order method on the grid $X = 0.5$ (for the fourth order method the shapes of curves are similar)

NUMERICAL EXAMPLES

To solve the problem (11) we have assumed $M = 97.5$ for the second order methods, and $P = Q = 961.4$ for the fourth order ones. Applying the procedure described earlier we can obtain similar values (for the constant M – see Figure 6).

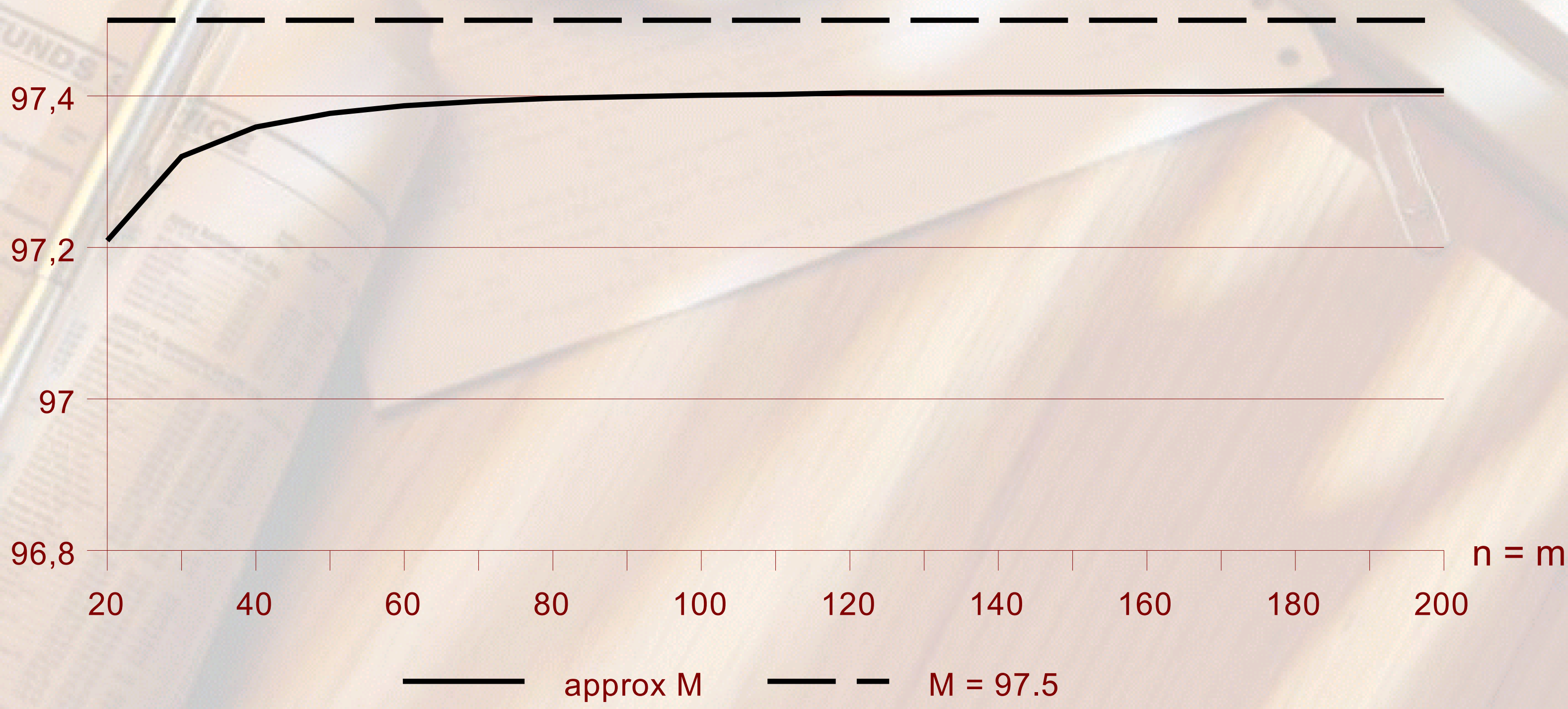

Figure 6. Approximations to the constant M for the problem (11)

CONCLUSIONS AND FURTHER STUDIES

- ★ Interval methods for solving partial-differential equation problems in floating-point interval arithmetic give solutions in the form of intervals which contain all possible numerical errors, i.e. representation, rounding and truncation errors.
- ★ The interval difference methods of fourth order are (of course) better than the methods of second order (give intervals with smaller widths).
- ★ The interval difference methods realized in directed floating--point interval atithmetic are longer in time (approximately 15%) than by the methods realized in proper one, but yield interval solutions with a little bit smaller widths.
- ★ Depending on the problem considered, the differences in widths may be decreasing or increasing in the number of mesh points, but in all cases the widths of intervals for directed interval arithmetic are a little bit smaller.

CONCLUSIONS AND FURTHER STUDIES

To have more valuable approximations for constants used in our methods, in further studies we plan to use the Nakao interval estimations to partial derivatives[*)]. Moreover, according to a special form of the system of (interval) linear equations that have to be solved, some more effective methods should also be taken into account.

We will also try to solve a generalized Poisson equation of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2}(x,y) + b(x,y)\frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y),$$

where

$$a(x,y)b(x,y) > 0,$$

with some boundary conditions, and to use other interval difference methods.

[*)] see, e.g., M. T. Nakao, On verified computations of solutions for nonlinear parabolic problems, *Nonlinear Theory and Its Applications*, IEICE 5 (3) (2014), 320-338.

Thank you for your attention

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