An Interval Difference Method of Second Order for Solving an Elliptic BVP

AndrzejMarciniak [1, 2], Małgorzata A. Jankowska [3], Tomasz Hoffmann [4]

[1] Institute of Computing Science, Poznan University of Technology, Poland
[2] Department of Computer Science, State University of Applied Sciences in Kalisz, Poland
[3] Institute of Applied Mechanics, Poznan University of Technology, Poland
[4] Poznan Supercomputing and Networking Center, Poland

13th International Conference on Parallel Processing and Applied Mathematics

Bialystok, Poland
September 8 – 11, 2019

Outline

- Introduction
- Boundary Value Problem for Elliptic Equations
- An Interval Difference Method
- Numerical Examples
- Conclusions

Introduction

It is well-known that floating-point arithmetic causes rounding errors, both for the representation of real numbers and for the result of operations. Applying approximate methods to solve problems on a computer we introduce also the error of methods (usually called the truncation errors).

Using interval methods realized in interval floating-point arithmetic we can obtain interval enclosures of solutions which are guaranteed to contain the actual solution.

Introduction

In this presentation we consider an interval difference method for more general elliptic equations with Dirichlet's boundary conditions than in our previous papers [1 – 6]. The generalization consists in taking into account some continuous functions about the second order partial derivatives and in adding a term $c(x, y) \cdot u(x, y)$ into equation, where $c(x, y)$ denotes also such a function.

---

[1] Hoffmann, T., Marciniak, A, Solving the Poisson Equation by an Interval Method of the Second Order, Computational Methods in Science and Technology 19 (1) (2013), 13–21.

[2] Hoffmann, T., Marciniak, A., Szyszka, B., Interval Versions of Central Difference Method for Solving the Poisson Equation in Proper and Directed Interval Arithmetic, Foundations of Computing and Decision Sciences 38 (3) (2013), 193–206.

[3] Hoffmann, T., Marciniak, A., Finding Optimal Numerical Solutions in Interval Versions of Central-Difference Method for Solving the Poisson Equation, Chapter 5 in: Data Analysis – Selected Problems (editors: M. Łatuszyńska, K. Nermend), Scientific Papers of the Polish Information Processing Society Scientific Council, Szczecin-Warsaw 2013, 79–88.

[4] Hoffmann, T., Marciniak, A., Solving the Generalized Poisson Equation in Proper and Directed Interval Arithmetic, Computational Methods in Science and Technology 22 (4) (2016), 225–232.

[5] Marciniak, A., An Interval Difference Method for Solving the Poisson Equation – the First Approach, Pro Dialog 24 (2008), 49–61.

[6] Marciniak, A., Hoffmann, T., Interval Difference Methods for Solving the Poisson Equation, in: Differential and Difference Equations with Applications (editors: S. Pinelas, T. Caraballo, P. Kloeden, J.R. Graef), Springer Proceedings in Mathematics & Statistics, vol. 230 (2018), 259–270.

Introduction

After a presentation of considered problem and our interval method, four numerical examples are given. These examples, like a number of other examples we have solved, show that the exact solutions belong to the interval enclosures obtained by our method.

Since, in our opinion, it is rather impossible to obtain a theoretical proof of this fact, the presentation can be treated as an experimental one.

Boundary value problem for elliptic equations

The well-known general form of elliptic partial differential equation is as follows:

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+2g(x,y)\frac{\partial^2 u}{\partial x\partial y}+b(x,y)\frac{\partial^2 u}{\partial y^2}$$
$$+2d(x,y)\frac{\partial u}{\partial x}+2e(x,y)\frac{\partial u}{\partial y}+c(x,y)u=f(x,y), \tag{1}$$

where $u = u(x, y)$, $0 \le x \le \alpha$, $0 \le y \le \beta$. The functions $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$, $d = d(x, y)$, $e = e(x, y)$, $f = f(x, y)$ and $g = g(x, y)$ are arbitrary continuous functions determined in the rectangle $\Omega = \{(x, y)\colon 0 \le x \le \alpha, 0 \le y \le \beta\}$ fulfilling in the interior of Ω the condition $a(x, y)b(x, y) - g^2(x, y) > 0$.

Boundary value problem for elliptic equations

For (1) we can consider the Dirichlet boundary conditions of the form

$$u\big|_{\Gamma} = \varphi(x, y) = \begin{cases} \varphi_1(y) & \text{for } x = 0, \\ \varphi_2(x) & \text{for } y = 0, \\ \varphi_3(y) & \text{for } x = \alpha, \\ \varphi_4(x) & \text{for } y = \beta, \end{cases} \tag{2}$$

where

$$\varphi_1(0) = \varphi_2(0), \quad \varphi_2(\alpha) = \varphi_3(0), \quad \varphi_3(\beta) = \varphi_4(\alpha), \quad \varphi_4(0) = \varphi_1(\beta),$$

and $\Gamma = \{(x, y)\colon\ x = 0, \alpha \text{ and } 0 \le y \le \beta \text{ or } 0 \le x \le \alpha \text{ and } y = 0, \beta\}$.

Boundary value problem for elliptic equations

If in (1) we take $a(x, y) = b(x, y) = 1$ and $c(x, y) = d(x, y) = e(x, y) =$ $= g(x, y) = 0$, then we have the following well-known Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y).$$

Interval difference methods for solving this equation with boundary conditions (2) we have presented in [1 – 3, 5, 6].

---

[1] Hoffmann, T., Marciniak, A, Solving the Poisson Equation by an Interval Method of the Second Order, Computational Methods in Science and Technology 19 (1) (2013), 13–21.
[2] Hoffmann, T., Marciniak, A., Szyszka, B., Interval Versions of Central Difference Method for Solving the Poisson Equation in Proper and Directed Interval Arithmetic, Foundations of Computing and Decision Sciences 38 (3) (2013), 193–206.
[3] Hoffmann, T., Marciniak, A., Finding Optimal Numerical Solutions in Interval Versions of Central-Difference Method for Solving the Poisson Equation, Chapter 5 in: Data Analysis – Selected Problems (editors: M. Łatuszyńska, K. Nermend), Scientific Papers of the Polish Information Processing Society Scientific Council, Szczecin-Warsaw 2013, 79–88.
[5] Marciniak, A., An Interval Difference Method for Solving the Poisson Equation – the First Approach, Pro Dialog 24 (2008), 49–61.
[6] Marciniak, A., Hoffmann, T., Interval Difference Methods for Solving the Poisson Equation, in: Differential and Difference Equations with Applications (editors: S. Pinelas, T. Caraballo, P. Kloeden, J.R. Graef), Springer Proceedings in Mathematics & Statistics, vol. 230 (2018), 259–270.

Boundary value problem for elliptic equations

The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + c(x, y)u = f(x, y)$$

is another special kind of elliptic equation of the form (1). In [7] we have constructed interval difference scheme for solving this equation with conditions (2) and compared with Nakao's method [8] based on Galerkin's approximation and finite elements.

---

[7] Marciniak, A., Nakao's Method and an Interval Difference Scheme of Second Order for Solving the Elliptic BVS, Computational Methods in Science and Technology 25 (2) (2019), 81–97.
[8] Nakao, M.T., A Numerical Approach to the Proof of Existence of Solutions for Elliptic Problems, Japan Journal of Applied Mathematics 5 (1988), 313–332.

Boundary value problem for elliptic equations

In this presentation we consider the elliptic differential equation of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} + c(x,y)u = f(x,y), \qquad (3)$$

in which

$$a(x,y)b(x,y) > 0$$

in the interior of rectangle Ω.

An interval difference method

Partitioning the interval $[0, \alpha]$ into n equal parts of width h and interval $[0, \beta]$ into m equal parts of width k provides a mean of placing a grid on the rectangle $[0, \alpha] \times [0, \beta]$ with mesh points (x_i, y_j), where $h = \alpha/n$, $k = \beta/m$. Assuming that the fourth order partial derivatives of u exist and using Taylor series in the variable x about x_i and in the variable y about y_j we can express the equation (3) at the points (x_i, y_j) as

$$a_{ij}\left[\delta_x^2 u_{ij} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j)\right] + b_{ij}\left[\delta_y^2 u_{ij} - \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)\right] + c_{ij}u_{ij} = f_{ij}, \tag{4}$$

where

$$\delta_x^2 u_{ij} = \frac{u_{i+1,\,j} - 2u_{ij} + u_{i-1,\,j}}{h^2}, \quad \delta_y^2 u_{ij} = \frac{u_{i,\,j+1} - 2u_{ij} + u_{i,\,j-1}}{k^2},$$

$$i = 1, 2, \ldots, n-1;\ j = 1, 2, \ldots, m-1,$$

$$v_{ij} = v(x_i, y_j) \text{ for } v \in \{u, a, b, c, f\},$$

and where $\xi_i \in (x_{i-1}, x_{i+1})$, $\eta_j \in (y_{j-1}, y_{j+1})$ are intermediate points, and the boundary conditions (2) as

$$
\begin{aligned}
u(0, y_j) &= \varphi_1(y_j) \text{ for } j = 0, 1, \ldots, m,\\
u(x_i, 0) &= \varphi_2(x_i) \text{ for } i = 1, 2, \ldots, n-1,\\
u(\alpha, y_j) &= \varphi_3(y_j) \text{ for } j = 0, 1, \ldots, m,\\
u(x_i, \beta) &= \varphi_4(x_i) \text{ for } i = 1, 2, \ldots, n-1.
\end{aligned}
\tag{5}
$$

An interval difference method

Differentiating (3) with respect to x and y, we have

$$a\frac{\partial^3 u}{\partial x^3}=\frac{\partial f}{\partial x}-\frac{\partial a}{\partial x}\frac{\partial^2 u}{\partial x^2}-\frac{\partial b}{\partial x}\frac{\partial^2 u}{\partial y^2}-b\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial c}{\partial x}u-c\frac{\partial u}{\partial x},$$

$$b\frac{\partial^3 u}{\partial y^3}=\frac{\partial f}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial^2 u}{\partial x^2}-a\frac{\partial^3 u}{\partial x^2\partial y}-\frac{\partial b}{\partial y}\frac{\partial^2 u}{\partial y^2}-\frac{\partial c}{\partial y}u-c\frac{\partial u}{\partial y},\qquad (6)$$

and differentiating again with respect to x and y, we get

$$a\frac{\partial^4 u}{\partial x^4}=\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 a}{\partial x^2}\frac{\partial^2 u}{\partial x^2}-2\frac{\partial a}{\partial x}\frac{\partial^3 u}{\partial x^3}-\frac{\partial^2 b}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-2\frac{\partial b}{\partial x}\frac{\partial^3 u}{\partial x\partial y^2}-b\frac{\partial^4 u}{\partial x^2\partial y^2}$$
$$-\frac{\partial^2 c}{\partial x^2}u-2\frac{\partial c}{\partial x}\frac{\partial u}{\partial x}-c\frac{\partial^2 u}{\partial x^2},\qquad (7)$$

$$b\frac{\partial^4 u}{\partial y^4}=\frac{\partial^2 f}{\partial y^2}-\frac{\partial^2 a}{\partial y^2}\frac{\partial^2 u}{\partial x^2}-2\frac{\partial a}{\partial y}\frac{\partial^3 u}{\partial x^2\partial y}-a\frac{\partial^4 u}{\partial x^2\partial y^2}-\frac{\partial^2 b}{\partial y^2}\frac{\partial^2 u}{\partial y^2}-2\frac{\partial b}{\partial y}\frac{\partial^3 u}{\partial y^3}$$
$$-\frac{\partial^2 c}{\partial y^2}u-2\frac{\partial c}{\partial y}\frac{\partial u}{\partial y}-c\frac{\partial^2 u}{\partial y^2}.$$

Taking into account in (7) the relations (6), we obtain

$$a\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} - \frac{2}{a}\frac{\partial a}{\partial x}\frac{\partial f}{\partial x} - \left[\frac{\partial^2 a}{\partial x^2} - \frac{2}{a}\left(\frac{\partial a}{\partial x}\right)^2 + c\right]\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial^2 b}{\partial x^2} - \frac{2}{a}\frac{\partial a}{\partial x}\frac{\partial b}{\partial x}\right)\frac{\partial^2 u}{\partial y^2}$$
$$-2\left(\frac{\partial b}{\partial x} - \frac{b}{a}\frac{\partial a}{\partial x}\right)\frac{\partial^3 u}{\partial x \partial y^2} - b\frac{\partial^4 u}{\partial x^2 \partial y^2} - \left(\frac{\partial^2 c}{\partial x^2} - \frac{2}{a}\frac{\partial a}{\partial x}\frac{\partial c}{\partial x}\right)u - 2\left(\frac{\partial c}{\partial x} - \frac{c}{a}\frac{\partial a}{\partial x}\right)\frac{\partial u}{\partial x} \tag{8}$$

and

$$b\frac{\partial^4 u}{\partial y^4} = \frac{\partial^2 f}{\partial y^2} - \frac{2}{b}\frac{\partial b}{\partial y}\frac{\partial f}{\partial y} - \left(\frac{\partial^2 a}{\partial y^2} - \frac{2}{b}\frac{\partial a}{\partial y}\frac{\partial b}{\partial y}\right)\frac{\partial^2 u}{\partial x^2} - \left[\frac{\partial^2 b}{\partial y^2} - \frac{2}{b}\left(\frac{\partial b}{\partial y}\right)^2 + c\right]\frac{\partial^2 u}{\partial y^2}$$
$$-2\left(\frac{\partial a}{\partial y} - \frac{a}{b}\frac{\partial b}{\partial y}\right)\frac{\partial^3 u}{\partial x^2 \partial y} - a\frac{\partial^4 u}{\partial x^2 \partial y^2} - \left(\frac{\partial^2 c}{\partial y^2} - \frac{2}{b}\frac{\partial b}{\partial y}\frac{\partial c}{\partial y}\right)u - 2\left(\frac{\partial c}{\partial y} - \frac{c}{b}\frac{\partial b}{\partial y}\right)\frac{\partial u}{\partial y}. \tag{9}$$

The equation (8) should be considered at (ξ_i, y_j) and the equation (9) – at (x_i, η_j).

An interval difference method

It is obvious that

$$b(\xi_i, y_j) = b_{ij} + O(h), \quad c(\xi_i, y_j) = c_{ij} + O(h), \quad \frac{1}{a(\xi_i, y_j)} = \frac{1}{a_{ij}} + O(h),$$

$$\frac{\partial^p v}{\partial x^p}(\xi_i, y_j) = \frac{\partial^p v}{\partial x^p}(x_i, y_j) + O(h) = \frac{\partial^p v_{ij}}{\partial x^p} + O(h),$$

(10)

$$a(x_i, \eta_j) = a_{ij} + O(k), \quad c(x_i, \eta_j) = c_{ij} + O(k), \quad \frac{1}{b(x_i, \eta_j)} = \frac{1}{b_{ij}} + O(k),$$

$$\frac{\partial^p v}{\partial y^p}(x_i, \eta_j) = \frac{\partial^p v}{\partial y^p}(x_i, y_j) + O(k) = \frac{\partial^p v_{ij}}{\partial x^p} + O(k)$$

for $p = 1, 2$ and $v = a, b, c$.

Moreover, we have

$$
\begin{aligned}
&\frac{\partial u}{\partial x}(\xi_i, y_j) = \frac{\partial u}{\partial x}(x_i, y_j) + O(h) = \delta_x u_{ij} + O(h),\\
&\frac{\partial^2 u}{\partial x^2}(\xi_i, y_j) = \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + O(h) = \delta_x^2 u_{ij} + O(h),\\
&\frac{\partial^2 u}{\partial y^2}(\xi_i, y_j) = \frac{\partial^2 u}{\partial y^2}(x_i, y_j) + O(h) = \delta_y^2 u_{ij} + O(k^2) + O(h),\\
&\frac{\partial u}{\partial y}(x_i, \eta_j) = \frac{\partial u}{\partial y}(x_i, y_j) + O(k) = \delta_y u_{ij} + O(k),\\
&\frac{\partial^2 u}{\partial y^2}(x_i, \eta_j) = \frac{\partial^2 u}{\partial y^2}(x_i, y_j) + O(k) = \delta_y^2 u_{ij} + O(k),\\
&\frac{\partial^2 u}{\partial x^2}(x_i, \eta_j) = \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + O(h) = \delta_x^2 u_{ij} + O(h^2) + O(k),
\end{aligned}
\tag{11}
$$

where

$$
\delta_x u_{ij} = \frac{u_{i+1,\,j} - u_{i-1,\,j}}{2h}, \quad \delta_y u_{ij} = \frac{u_{i,\,j+1} - u_{i,\,j-1}}{2k}.
$$

An interval difference method

Substituting (10) and (11) into (8) and (9), and then substituting the resulting formulas into (4), after some transformations we finally obtain

$$\begin{aligned}
&\left(\frac{w_{1ij}}{h^2}-\frac{w_{3ij}}{2h}\right)u_{i-1,\,j}+\left(\frac{w_{2ij}}{k^2}-\frac{w_{4ij}}{2k}\right)u_{i,\,j-1}\\
&-\left(2\frac{w_{1ij}}{h^2}+2\frac{w_{2ij}}{k^2}-w_{5ij}-w_{6ij}-c_{ij}\right)u_{ij}\\
&+\left(\frac{w_{2ij}}{k^2}+\frac{w_{4ij}}{2k}\right)u_{i,\,j+1}+\left(\frac{w_{1ij}}{h^2}-\frac{w_{3ij}}{2h}\right)u_{i+1,\,j}\\
&=f_{ij}+w_{7ij}+O(h^3)+O(k^3)+O(h^2k^2),
\end{aligned} \tag{12}$$

where

$$w_{1ij}=a_{ij}+\frac{h^2}{12}\left[\frac{\partial^2 a_{ij}}{\partial x^2}-\frac{2}{a_{ij}}\left(\frac{\partial a_{ij}}{\partial x}\right)^2+c_{ij}\right]+\frac{k^2}{12}\left(\frac{\partial^2 a_{ij}}{\partial y^2}-\frac{2}{b_{ij}}\frac{\partial a_{ij}}{\partial y}\frac{\partial b_{ij}}{\partial y}\right),$$

$$w_{2ij}=b_{ij}+\frac{h^2}{12}\left(\frac{\partial^2 b_{ij}}{\partial x^2}-\frac{2}{a_{ij}}\frac{\partial a_{ij}}{\partial x}\frac{\partial b_{ij}}{\partial x}\right)+\frac{k^2}{12}\left[\frac{\partial^2 b_{ij}}{\partial y^2}-\frac{2}{b_{ij}}\left(\frac{\partial b_{ij}}{\partial y}\right)^2+c_{ij}\right],$$

$$w_{3ij}=\frac{h^2}{6}\left(\frac{\partial c_{ij}}{\partial x}-\frac{c_{ij}}{a_{ij}}\frac{\partial a_{ij}}{\partial x}\right),\quad w_{4ij}=\frac{k^2}{6}\left(\frac{\partial c_{ij}}{\partial y}-\frac{c_{ij}}{b_{ij}}\frac{\partial b_{ij}}{\partial y}\right),$$

$$w_{5ij}=\frac{h^2}{12}\left(\frac{\partial^2 c_{ij}}{\partial x^2}-\frac{2}{a_{ij}}\frac{\partial a_{ij}}{\partial x}\frac{\partial c_{ij}}{\partial x}\right),\quad w_{6ij}=\frac{k^2}{12}\left(\frac{\partial^2 c_{ij}}{\partial y^2}-\frac{2}{b_{ij}}\frac{\partial b_{ij}}{\partial y}\frac{\partial c_{ij}}{\partial y}\right),$$

An interval difference method

$$
\begin{aligned}
w_{7ij} = {} & \frac{h^2}{12}\left[\frac{\partial^2 f}{\partial x^2}(\xi_i, y_j) - \frac{2}{a_{ij}}\frac{\partial a_{ij}}{\partial x}\frac{\partial f}{\partial x}(\xi_i, y_j)\right. \\
& \left. - 2\left(\frac{\partial b_{ij}}{\partial x} - \frac{b_{ij}}{a_{ij}}\frac{\partial a_{ij}}{\partial x}\right)\frac{\partial^3 u}{\partial x \partial y^2}(\xi_i, y_j) - b_{ij}\frac{\partial^4 u}{\partial x^2 \partial y^2}(\xi_i, y_j)\right] \\
& + \frac{k^2}{12}\left[\frac{\partial^2 f}{\partial y^2}(x_i, \eta_j) - \frac{2}{b_{ij}}\frac{\partial b_{ij}}{\partial y}\frac{\partial f}{\partial y}(x_i, \eta_j)\right. \\
& \left. - 2\left(\frac{\partial a_{ij}}{\partial y} - \frac{a_{ij}}{b_{ij}}\frac{\partial b_{ij}}{\partial y}\right)\frac{\partial^3 u}{\partial x^2 \partial y}(x_i, \eta_j) - a_{ij}\frac{\partial^4 u}{\partial x^2 \partial y^2}(x_i, \eta_j)\right].
\end{aligned}
$$

From (12) we can obtain an interval method.

Let us assume that

$$\left|\frac{\partial^4 u}{\partial x^2 \partial y^2}(x, y)\right| \le M, \quad \left|\frac{\partial^3 u}{\partial x^2 \partial y}(x, y)\right| \le P, \quad \left|\frac{\partial^3 u}{\partial x \partial y^2}(x, y)\right| \le Q$$

for all (x, y) in Ω, and let $\Psi_1(X, Y)$, $\Psi_2(X, Y)$, $\Xi_1(X, Y)$, $\Xi_2(X, Y)$ denote interval extensions of

$$\frac{\partial f}{\partial x}(x, y), \ \frac{\partial^2 f}{\partial x^2}(x, y), \ \frac{\partial f}{\partial y}(x, y), \ \frac{\partial^2 f}{\partial y^2}(x, y),$$

respectively.

Then

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y) \in [-M, M], \quad \frac{\partial^3 u}{\partial x^2 \partial y}(x,y) \in [-P, P], \quad \frac{\partial^3 u}{\partial x \partial y^2}(x,y) \in [-Q, Q]$$

for each (x, y), and

$$\frac{\partial f}{\partial x}(\xi_i, y_j) \in \Psi_1(X_i + [-h,h], Y_j), \quad \frac{\partial^2 f}{\partial x^2}(\xi_i, y_j) \in \Psi_2(X_i + [-h,h], Y_j),$$

$$\frac{\partial f}{\partial y}(x_i, \eta_j) \in \Xi_1(X_i, Y_j + [-k,k]), \quad \frac{\partial^2 f}{\partial y^2}(x_i, \eta_j) \in \Xi_2(X_i, Y_j + [-k,k]),$$

since $\xi_i \in (x_i - h, x_i + h)$ and $\eta_j \in (y_j - k, y_j + k)$.

An interval difference method

Thus, we have $w_{7ij} \in W_{7ij}$, where

$$W_{7ij} = \frac{h^2}{12}\left\{\Psi_2(X_i + [-h,h], Y_j) - \frac{2}{A_{ij}} D_x A_{ij} \Psi_1(X_i + [-h,h], Y_j)\right.$$
$$\left. - 2\left(D_x B_{ij} - \frac{B_{ij}}{A_{ij}} D_x A_{ij}\right)[-Q,Q] - B_{ij}[-M,M]\right\} \quad (13)$$
$$+ \frac{k^2}{12}\left\{\Xi_2(X_i, Y_j + [-k,k]) - \frac{2}{B_{ij}} D_y B_{ij} \Xi_1(X_i, Y_j + [-k,k])\right.$$
$$\left. - 2\left(D_y A_{ij} - \frac{A_{ij}}{B_{ij}} D_y B_{ij}\right)[-P,P] - A_{ij}[-M,M]\right\},$$

and where V_{ij} and $D_z V_{ij}$ for $V \in \{A, B\}$ and $z \in \{x, y\}$ denote interval extensions of v_{ij} and $\partial v_{ij}/\partial z$ for $v \in \{a, b\}$, respectively.

An interval difference method

If we denote interval extensions of f_{ij}, c_{ij} and w_{pij} by F_{ij}, C_{ij} and W_{pij}, respectively ($p = 1, 2, \ldots, 6$), then from the above considerations and (12) it follows an interval method of the form

$$\begin{aligned}
&\left(\frac{W_{1ij}}{h^2} - \frac{W_{3ij}}{2h}\right) U_{i-1,\,j} + \left(\frac{W_{2ij}}{k^2} - \frac{W_{4ij}}{2k}\right) U_{i,\,j-1} \\
&- \left(\frac{2W_{1ij}}{h^2} + \frac{2W_{2ij}}{k^2} - W_{5ij} - W_{6ij} - C_{ij}\right) U_{ij} \\
&+ \left(\frac{W_{2ij}}{k^2} + \frac{W_{4ij}}{2k}\right) U_{i,\,j+1} + \left(\frac{W_{1ij}}{h^2} + \frac{W_{3ij}}{2h}\right) U_{i+1,\,j} \\
&= F_{ij} + W_{7ij} + [-\delta, \delta], \quad i = 1, 2, \ldots, n-1, \quad j = 1, 2, \ldots, m-1,
\end{aligned} \tag{14}$$

where the interval $[-\delta, \delta]$, called the δ-extension, represents $O(h^3) + O(k^3) + O(h^2k^2)$, and where

$$U_{0j} = \Phi_1(Y_j), \quad U_{i0} = \Phi_2(X_i), \quad U_{nj} = \Phi_3(Y_j), \quad U_{im} = \Phi_4(X_i),$$
$$j = 0, 1, \ldots, m, \quad i = 1, 2, \ldots, n-1. \tag{15}$$

Here, $\Phi_1(Y)$, $\Phi_2(X)$, $\Phi_3(Y)$ and $\Phi_4(X)$ denote interval extensions of $\varphi_1(y)$, $\varphi_2(x)$, $\varphi_3(y)$ and $\varphi_4(x)$, respectively.

The system of linear interval equations (14) with (15), with unknowns U_{ij}, can be solved in conventional (proper) floating-point interval arithmetic, because all intervals are proper.

It should be added a remark concerning the constants M, P and Q occurring in (13). If nothing can be concluded about M, P and Q from physical or technical properties or characteristics of the problem considered, we proposed to find these constants taking into account that

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x_i, y_j) = \lim_{h \to 0} \lim_{k \to 0} \left(\frac{u_{i-1,\,j-1} + u_{i-1,\,j+1} + u_{i+1,\,j-1} + u_{i+1,\,j+1}}{h^2 k^2} + \frac{4u_{ij} - 2(u_{i-1,\,j} + u_{i,\,j-1} + u_{i,\,j+1} + u_{i+1,\,j}}{h^2 k^2} \right),$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \lim_{h \to 0} \lim_{k \to 0} \left(\frac{u_{i-1,\,j+1} - u_{i-1,\,j-1} - 2(u_{i1,\,j+1} - u_{i1,\,j-1}) + u_{i+1,\,j+1} - u_{i+1,\,j-1}}{2h^2 k} \right),$$

$$\frac{\partial^3 u}{\partial x \partial y^2} = \lim_{h \to 0} \lim_{k \to 0} \left(\frac{u_{i+1,\,j-1} - u_{i-1,\,j-1} - 2(u_{i+1,\,j} - u_{i-1,\,j} + u_{i+1,\,j+1} - u_{i-1,\,j+1}}{2hk^2} \right).$$

We can calculate

$$M_{nm} = \frac{1}{h^2k^2} \max_{\substack{i=1,2,\dots,n-1 \\ j=1,2,\dots,m-1}} \left| u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} \right.$$

$$\left. + 4u_{ij} - 2(u_{i-1,j} + u_{i,j-1} + u_{i,j+1} + u_{i+1,j}) \right|,$$

$$P_{nm} = \frac{1}{2h^2k} \max_{\substack{i=1,2,\dots,n-1 \\ j=1,2,\dots,m-1}} \left| u_{i-1,j+1} - u_{i-1,j-1} - 2(u_{i,j+1} - u_{i,j-1}) \right.$$

$$\left. + u_{i+1,j+1} - u_{i+1,j-1} \right|,$$

$$Q_{nm} = \frac{1}{2hk^2} \max_{\substack{i=1,2,\dots,n-1 \\ j=1,2,\dots,m-1}} \left| u_{i+1,j-1} - u_{i-1,j-1} - 2(u_{i+1,j} - u_{i-1,j}) \right.$$

$$\left. + u_{i+1,j+1} - u_{i-1,j+1} \right|.$$

where u_{ij} are obtained by a conventional method for a variety of n and m.

Then we can plot M_{nm}, P_{nm} and Q_{nm} against different n and m. The constants M, P and Q can be easy determined from the obtained graphs, since

$$\lim_{\substack{n \to \infty \\ m \to \infty}} M_{nm} \leq M, \quad \lim_{\substack{n \to \infty \\ m \to \infty}} P_{nm} \leq P, \quad \lim_{\substack{n \to \infty \\ m \to \infty}} Q_{nm} \leq Q.$$

Numerical examples

In the examples presented we have used our own implementation of floating-point interval arithmetic written in Delphi Pascal. This implementation has been written as a unit called *IntervalArithmetic32and64*, which current version may be found in [9]. The program written in Delphi Pascal for the example considered one can find in [10] and [11]. We have run this program on Lenovo® Z51 computer with Intel® Core i7 2.4 GHz processor.

---

[9] Marciniak, A., Interval Arithmetic Unit (2016), URL http//www.cs.put.poznan.pl/amarciniak/ IAUnits/IntervalArithmetic32and64.pas.
[10] Marciniak, A., Delphi Pascal Programs for Elliptic Boundary Value Problem (2019), URL http// www.cs.put.poznan.pl/amarciniak/IDM- EllipticEqn-Example
[11] Marciniak, A., Delphi Pascal Programs for Nakao and Interval Difference Methods for Solving the Elliptic BVP (2019), URL http://www.cs.put.poznan.pl/amarciniak/NIDM-EllipticBVP-Examples

Example 1

Let $\Omega = [0,1] \times [0, 1]$ and consider the following problem:

$$x^2 \sin \pi y \frac{\partial^2 u}{\partial x^2} + y^2 \sin \pi x \frac{\partial^2 u}{\partial y^2} - xyu$$

$$= xy \exp(xy)\left[2xy \sin\frac{\pi(x+y)}{2} \cos\frac{\pi(x-y)}{2} - 1\right],$$

$$\varphi_1(y) = 1, \quad \varphi_2(x) = 1, \quad \varphi_3(y) = \exp(y), \quad \varphi_4(x) = \exp(x).$$

This problem has the exact solution $u(x, y) = \exp(xy)$. Since the exact solution is known, we can calculate the constants M, P and Q and take $M = 19.03$, $P = Q = 8.16$. These constants can be also estimates from the graphs presented in Fig. 1 and 2 (the method still succeeds for less accurate bounds).

Numerical examples

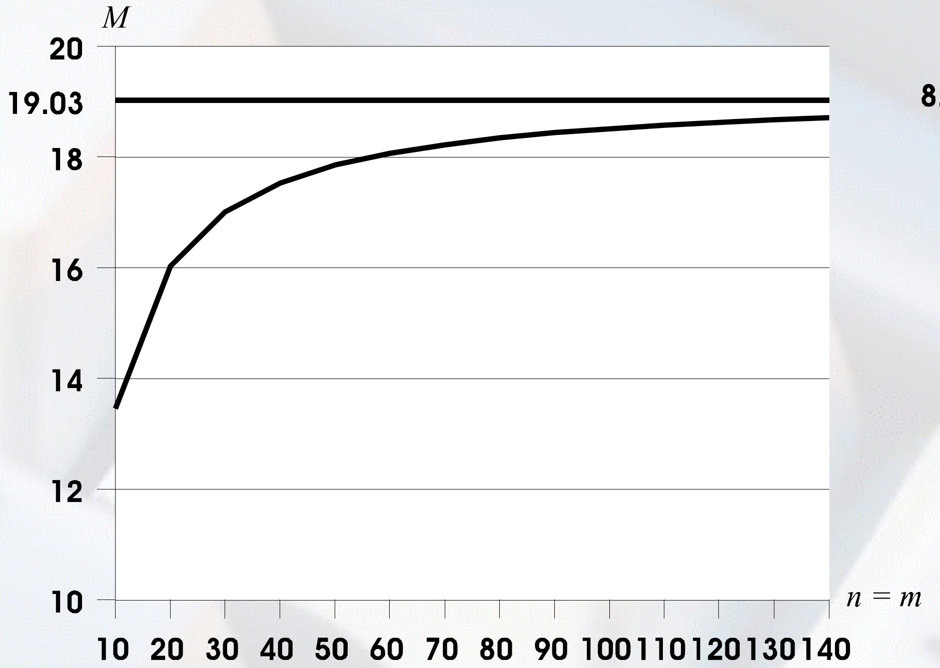

Figure 1. Estimations of M

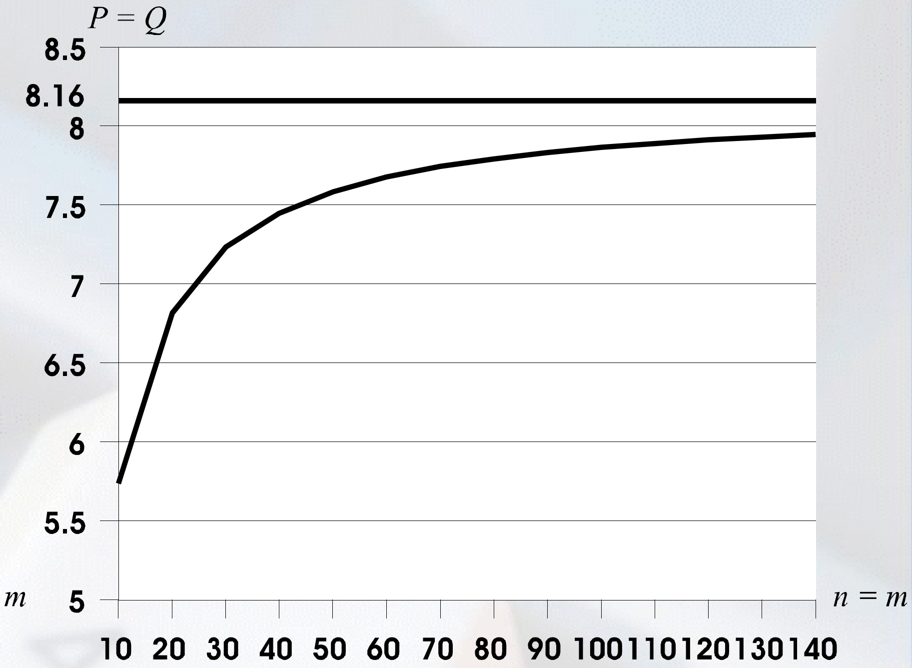

Figure 2. Estimations of P and Q

For $h = k = 0.01$, i.e., $n = m = 100$, and $\delta = 10^{-6}$, using an interval version of LU decomposition, after 6 minutes we have obtained by our program [10] the results presented in Table 1.

Note that using the interval version of full Gauss elimination (with pivoting) we need about 200 days (!) of CPU time to obtain such results.

---

[10] Marciniak, A., Delphi Pascal Programs for Elliptic Boundary Value Problem (2019), URL http// www.cs.put.poznan.pl/amarciniak/IDM- EllipticEqn-Example

Numerical examples

Table 1. Enclosures of the exact solution obtained by the method (14) – (15)

(x_i, y_j)	U_{ij}	*Width*
(0.1, 0.5)	[1.0509815441511615E+0000, 1.0515636639134466E+0000]	$\approx 5.82 \cdot 10^{-4}$
	exact ≈ 1.0512710963760240E+0000	
(0.3, 0.5)	[1.1616066450304161E+0000, 1.1620625933013910E+0000]	$\approx 4.56 \cdot 10^{-4}$
	exact ≈ 1.1618342427282831E+0000	
(0.5, 0.1)	[1.0509815441511614E+0000, 1.0515636639134467E+0000]	$\approx 5.82 \cdot 10^{-4}$
	exact ≈ 1.0512710963760240E+0000	
(0.5, 0.3)	[1.1616066450304160E+0000, 1.1620625933013911E+0000]	$\approx 4.56 \cdot 10^{-4}$
	exact ≈ 1.1618342427282831E+0000	
(0.5, 0.5)	[1.2838256641451216E+0000, 1.2842221958365290E+0000]	$\approx 3.97 \cdot 10^{-4}$
	exact ≈ 1.2840254166877415E+0000	
(0.5, 0.7)	[1.4189191705563123E+0000, 1.4192105027251595E+0000]	$\approx 2.91 \cdot 10^{-4}$
	exact ≈ 1.4190675485932573E+0000	
(0.5, 0.9)	[1.5682528588111690E+0000, 1.5683682637095705E+0000]	$\approx 1.15 \cdot 10^{-4}$
	exact ≈ 1.5683121854901688E+0000	
(0.7, 0.5)	[1.4189191705563122E+0000, 1.4192105027251595E+0000]	$\approx 2.91 \cdot 10^{-4}$
	exact ≈ 1.4190675485932573E+0000	
(0.9, 0.5)	[1.5682528588111689E+0000, 1.5683682637095705E+0000]	$\approx 1.15 \cdot 10^{-4}$
	exact ≈ 1.5683121854901688E+0000	

One can observe that for each (x_i, y_j) the exact solution is within the interval enclosures obtained. It should be added that CPU time grows significantly for greater values of n and m.

Example 2 [8, pp. 330 – 331]

Let us consider the following problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \pi u = (1 - 2\pi)\sin \pi x \sin \pi y \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \Gamma, \tag{16}$$

where $\Omega = [0, 1] \times [0, 1]$. In this problem we have $a(x, y) = b(x, y) = 1$ and $c(x, y) = \pi$. The exact solution of (16) is of the form (see Fig. 3)

$$u(x, y) = \frac{1}{\pi}\sin \pi x \sin \pi y. \tag{17}$$

[8] Nakao, M.T., A Numerical Approach to the Proof of Existence of Solutions for Elliptic Problems, Japan Journal of Applied Mathematics 5 (1988), 313–332.

Numerical examples

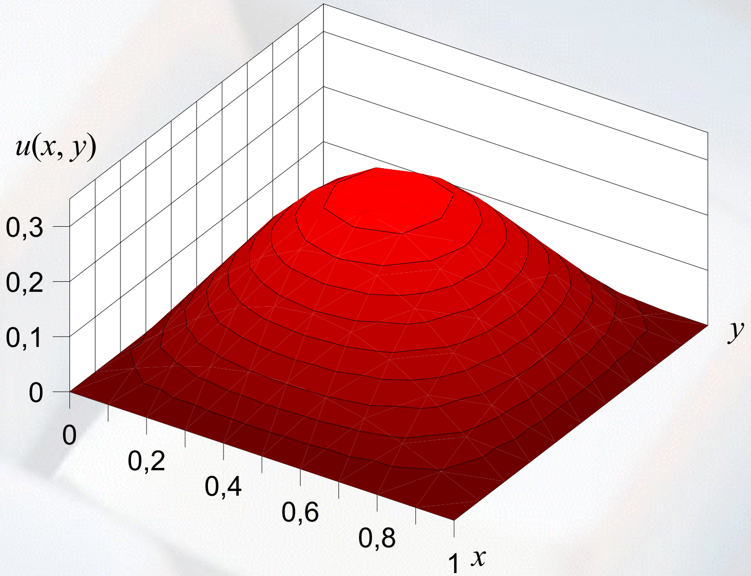

Figure 3. A graph of solution (34) to the problem (33)

Using Nakao's method*) and taking (as Nakao) $h = 0.1$, i.e., $n = 10$, the initial values from Galerkin's approximation, the stopping and extension parameters $\varepsilon = \delta = 10^{-8}$, we obtain after $N = 10$ iterations the results presented in Table 2. In the same table we also present the results obtained by Nakao.

One can observe that our intervals are thinness than those presented by Nakao in his original paper. Moreover, it should be added that Nakao obtained his results after $N = 16$ iterations, i.e., in a greater number of iterations than in our implementation.

*) The Nakao method is an iteration method based on Galerkin's approximation and finite element method known from conventional theory for solving elliptic problems.

Numerical examples

Table 2. Enclosures of solution (17) to the problem (16) obtained by Nakao's method
(N – intervals presented in [8])

(i, j)	$U(X_i, Y_j)$	$Width \times 10^3$
(1, 1)	[2.6895096700530953E−0002, 3.2427221086186715E−0002]	
	≈ [0.0268950, 0.0324273]	≈ 5.5
	N [0.0152852, 0.0459134]	≈ 30.6
	exact ≈ 0.0303959	
(1, 5)	[9.0624935701623463E−0002, 1.0360043786745875E−0001]	
	≈ [0.0906249, 0.1036005]	≈ 13.0
	N [0.0641884, 0.1338544]	≈ 69.7
	exact ≈ 0.0983632	
(1, 9)	[2.7580990032784303E−0002, 3.3135690195121944E−0002]	
	≈ [0.0275809, 0.0331356]	≈ 5.6
	N [0.0152852, 0.0459134]	≈ 30.6
	exact ≈ 0.0303959	
(2, 1)	[5.2119428179302382E−0002, 6.1187858075021614E−0002]	
	≈ [0.0521194, 0.0611879]	≈ 9.1
	N [0.0335920, 0.0828146]	≈ 49.2
	exact ≈ 0.0578164	
(2, 5)	[1.7346343896287746E−0002, 1.9597642240178172E−0002]	
	≈ [0.1734634, 0.1959765]	≈ 22.5
	N [0.1278847, 0.2488152]	≈ 120.9
	exact ≈ 0.1870979	
(2, 9)	[5.2962951964144576E−0002, 6.2056169423667783E−0002]	
	≈ [0.0529629, 0.0620562]	≈ 9.1
	N [0.0335920, 0.0828146]	≈ 49.2
	exact ≈ 0.0578164	
(3, 1)	[7.2554266253714074E−0002, 8.3864670393438673E−0002]	
	≈ [0.0725542, 0.0838647]	≈ 11.3
	N [0.0496244, 0.1105956]	≈ 61.0
	exact ≈ 0.0795775	

c
o
m
p
a
r
e

Numerical examples

Two reasons may cause that in our implementation we have obtained better enclosures: we have used the exact integrals in the method, while probably in [8] some quadratures have been used (it causes to come into being some additional errors – errors of quadratures), and in our interval calculations we have taken advantage of Delphi Pascal *Extended* type, which is more precise than *Double* type used probably in [8] (this have rather small influence on the results).

The Nakao method gives validated solutions, but in our experimental interval difference method the obtained enclosures (intervals) are thiness, and – of course – all intervals contain the exact solution at the mesh points (see Table 3).

In our interval difference method we have taken $M = 62.02$ (since the exact solution is known). The values of constants P and Q are unnecessary since $a = b = 1$, and hence $\partial a/\partial x =$ $= \partial a/\partial y = \partial b/\partial x = \partial b/\partial y = 0$. Moreover, we assume in our method that $\delta = 0.001$, and – according to the boundary conditions – $U_{0j} = U_{i0} = U_{10,j} = U_{i,10} = 0$ ($j = 0, 1, \ldots, 10$; $i = 1, 2, \ldots, 9$).

[8] Nakao, M.T., A Numerical Approach to the Proof of Existence of Solutions for Elliptic Problems, Japan Journal of Applied Mathematics 5 (1988), 313–332.

Numerical examples

Table 3. Enclosures of solution (17) to the problem (16) obtained by the method (14) – (15)
(N – Nakao's method)

(i, j)	$U(X_i, Y_j)$	$Width \times 10^3$
(1, 1)	[2.8413489793818334E−0002, 3.2077265656902749E−0002]	
	≈ [0.0284134, 0.0320773]	≈ 3.7
	exact ≈ 0.0303959	N ≈ 5.5
(1, 5)	[9.3435235810096799E−0002, 1.0258291837474664E−0001]	
	≈ [0.0934352, 0.1025830]	≈ 9.1
	exact ≈ 0.0983632	N ≈ 13.0
(1, 9)	[2.8373268255628631E−0002, 3.2303880411274499E−0002]	
	≈ [0.0283732, 0.0323039]	≈ 3.9
	exact ≈ 0.0303959	N ≈ 5.6
(2, 1)	[5.4472248034826312E−0002, 6.0608703745006735E−0002]	
	≈ [0.0544722, 0.0606088]	≈ 6.1
	exact ≈ 0.0578164	N ≈ 9.1
(2, 5)	[1.7840742966978266E−0001, 1.9450823450046243E−0001]	
	≈ [0.1784074, 0.1945083]	≈ 16.1
	exact ≈ 0.1870979	N ≈ 22.5
(2, 9)	[5.4395742122844850E−0002, 6.1039750622674490E−0002]	
	≈ [0.0543957, 0.0610398]	≈ 6.6
	exact ≈ 0.0578164	N ≈ 9.1
(3, 1)	[7.5335356959410713E−0002, 8.3113638230475435E−0002]	
	≈ [0.0753353, 0.0831137]	≈ 7.8
	exact ≈ 0.0795775	N ≈ 11.3

compare

Example 3 [8, pp. 329 – 330]

For the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 20xyu = (1 - 2\pi)\sin \pi x \sin \pi y \ \text{ in } \ \Omega, \qquad (18)$$

$$u = 0 \ \text{ on } \ \Gamma,$$

where $\Omega = [0, 1] \times [0, 1]$, the exact solution is unknown.

Taking $h = 0.1$, the initial values from Galerkin's approximation, the stopping parameter $\varepsilon = 10^{-4}$ and the extension parameter $\delta = 10^{-3}$, in our implementation of Nakao's method we obtain after $N = 7$ iterations the results presented in Table 4. The Nakao results (published in [8]) obtained after $N = 10$ are also presented in the same table. As in Example 2 we can observe that our intervals are significant thinness and Nakao obtained his results after a greater number of iterations.

[8] Nakao, M.T., A Numerical Approach to the Proof of Existence of Solutions for Elliptic Problems, Japan Journal of Applied Mathematics 5 (1988), 313–332.

Numerical examples

Table 4. Enclosures of solution to the problem (18) obtained by Nakao's method (N – intervals presented in [8])

(i,j)		$U(X_i, Y_j)$	$Width \times 10^3$
(1, 1)		[2.542209692014740 1E−0002, 3.3990702176130446E−0002]	
	≈	[0.0254220, 0.0339908]	≈ 8.6
	N	[0.0044953, 0.0603775]	≈ 55.9
(1, 5)		[9.2513778055918294E−0002, 1.1033908384199032E−0001]	
	≈	[0.0925137, 0.1103391]	≈ 17.8
	N	[0.0066034, 0.2148457]	≈ 208.2
(1, 9)		[2.8693572427687149E−0002, 3.7601391162418901E−0002]	
	≈	[0.0286935, 0.0376014]	≈ 8.9
	N	[−0.0158443, 0.0880232]	≈ 103.9
(2, 1)		[5.0699038788017661E−0002, 6.3495793925095351E−0002]	
	≈	[0.0506990, 0.0634958]	≈ 12.8
	N	[0.0097055, 0.1147773]	≈ 105.1
(2, 5)		[1.8060879979601782E−0001, 2.1053874248141714E−0001]	
	≈	[0.1806087, 0.2105388]	≈ 29.9
	N	[0.0133514, 0.4145900]	≈ 401.2
(2, 9)		[5.7342513568695084E−0002, 7.0907900667225485E−0002]	
	≈	[0.0573425, 0.0709080]	≈ 13.6
	N	[−0.0269411, 0.1674148]	≈ 194.4
(3, 1)		[7.1830592564795797E−0002, 8.7386541698192201E−0002]	
	≈	[0.0718305, 0.0873866]	≈ 15.5
	N	[0.0124607, 0.1610751]	≈ 148.6

compare

In order to use our interval difference method we need to evaluate the constant M (as previously, the constants P and Q are unnecessary). Since the exact solution is unknown, we can calculate M_{nn} for different values of n. The obtained results are presented in Fig. 4. From this figure it follows that $M \approx 38.9$. For $h = 0.1$ and $\delta = 0.001$ from (14) – (15) we obtain enclosures of the exact solution presented in Table 5 (these enclosure are thinness than those presented in Table 4).

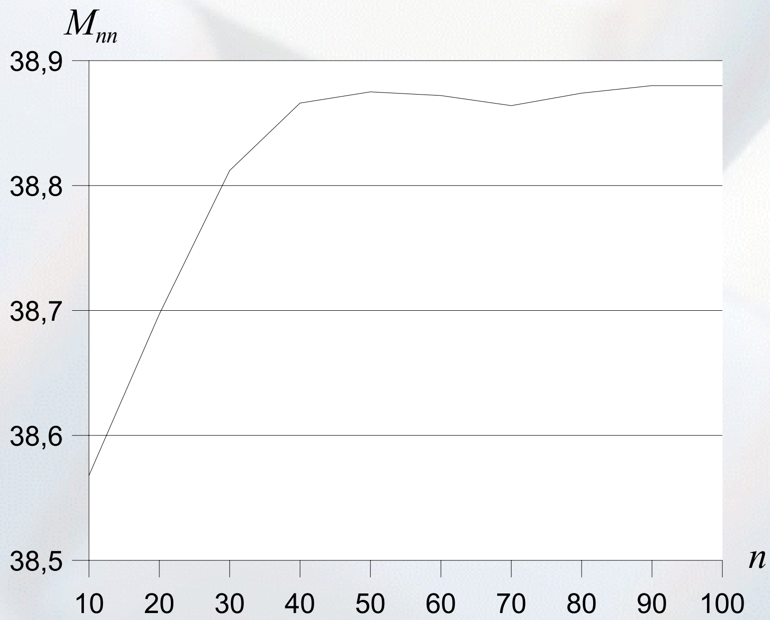

Figure 4. A graph for M_{nn} for n = 10, 20, ... , 100

Numerical examples

Table 5. Enclosures of solution to the problem (18) obtained by the method (14) – (15)
(N – Nakao's method)

(i, j)	$U(X_i, Y_j)$	$Width \times 10^3$
(1, 1)	[2.8997209867618227E−0002, 3.1544047883616199E−0002] ≈ [0.0289972, 0.0315441]	≈ 2.5 N ≈ 8.6
(1, 5)	[9.9185893244672646E−0002, 1.0604503147224523E−0001] ≈ [0.0991858, 0.1060451]	≈ 6.9 N ≈ 17.8
(1, 9)	[3.1726223260720911E−0002, 3.4740582521688013E−0002] ≈ [0.0317262, 0.0347406]	≈ 3.0 N ≈ 8.9
(2, 1)	[5.5847687322568013E−0002, 6.0189730374459988E−0002] ≈ [0.0558476, 0.0601898]	≈ 4.3 N ≈ 12.8
(2, 5)	[1.9176543915299421E−0001, 2.0413632819753807E−0001] ≈ [0.1917654, 0.2041364]	≈ 12.4 N ≈ 29.9
(2, 9)	[6.1877653239000653E−0002, 6.7170747737222414E−0002] ≈ [0.0618776, 0.0671708]	≈ 5.3 N ≈ 13.6
(3, 1)	[7.7943013254661869E−0002, 8.3553350213709836E−0002] ≈ [0.0779430, 0.0835534]	≈ 5.6 N ≈ 15.5

compare

Example 4

Let

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{1+x^2+y^2}\left[\frac{2+x^2+y^2}{\sqrt{1+x^2+y^2}}\cos\sqrt{1+x^2+y^2}\right.$$
$$\left.-(x^2+y^2)\sin\sqrt{1+x^2+y^2}\right] \text{ in } \Omega \qquad (19)$$

and

$$u|_\Gamma=\begin{cases}\sin\sqrt{1+y^2} & \text{for } x=0,\\ \sin\sqrt{1+x^2} & \text{for } y=0,\\ \sin\sqrt{65+y^2} & \text{for } x=8,\\ \sin\sqrt{17+x^2} & \text{for } y=4,\end{cases} \qquad (20)$$

where $\Omega=[0,8]\times[0,4]$. The Poisson equation (19) with Dirichlet's conditions (20) has the solution of the form (see Fig. 5)

$$u(x,y)=\sin\sqrt{1+x^2+y^2}. \qquad (21)$$

Numerical examples

The problem (19) – (20) cannot be solved by Nakao's method since $c = 0$ (see [7] for details). But using our interval method we can obtain enclosures of (21) at some mesh points.

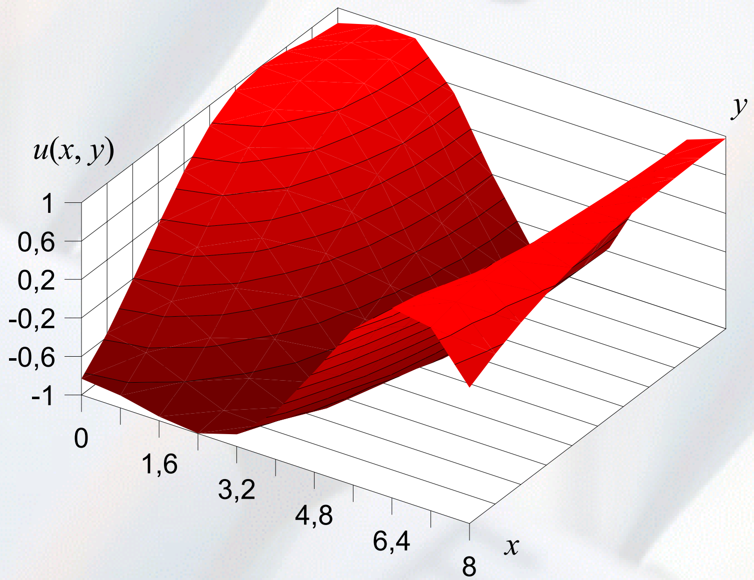

Figure 5. A graph of solution (21) to the problem (19) – (20)

Taking $n = 80$, $m = 40$ $(h = k = 0.1)$, $M = 1.382$ (this constant can be estimated since the exact solution is known), from our method in which the interval $[-\delta, \delta]$ does not occur (this interval never occurs for the Poisson equation), we have obtained enclosures of the exact solution presented in Table 6. As previously, all intervals contain the exact solution at the mesh points.

[7] Marciniak, A., Nakao's Method and an Interval Difference Scheme of Second Order for Solving the Elliptic BVS, Computational Methods in Science and Technology 25 (2) (2019), 81–97.

Numerical examples

Table 6. Enclosures of solution (21) to the problem (19) – (20) obtained by the method (14) – (15) (for selected *i* and *j*)

(i, j)	$U(X_i, Y_j)$	$Width \times 10^3$
(40, 8)	$[-8.7570448853762087E-0001, -8.6786852449020102E-0001]$ $\approx [-0.8757044, -0.8678686]$ exact ≈ -0.8715758	≈ 7.8
(40, 16)	$[-9.6473261912851148E-0001, -9.5273425661315874E-0001]$ $\approx [-0.9647326, -0.9527343]$ exact ≈ -0.9583239	≈ 12.0
(40, 24)	$[-1.0049018906555416E+0000, -9.9269471083206701E-0001]$ $\approx [-1.0049018, -0.9926948]$ exact ≈ -0.9982978	≈ 12.2
(40, 32)	$[-8.7882382725746346E-0001, -8.7056138147601539E-0001]$ $\approx [-0.8788238, -0.8705614]$ exact ≈ -0.8742991	≈ 8.3
(16, 20)	$[3.7736191041585845E-0001, 3.8651130223780119E-0001]$ $\approx [0.3773619, 0.3865114]$ exact ≈ 0.3820811	≈ 9.1
(32, 20)	$[-6.9697972969609782E-0001, -6.8473622032926368E-0001]$ $\approx [-0.6969797, -0.6847363]$ exact ≈ -0.6905517	≈ 12.2
(48, 20)	$[-8.4156500228698132E-0001, -8.2916731158025701E-0001]$ $\approx [-0.8415650, -0.8291674]$ exact ≈ -0.8348743	≈ 12.4
(64, 20)	$[4.7127662788337241E-0001, 4.8095589691916753E-0001]$ $\approx [0.4712766, 0.4809559]$ exact ≈ 0.4760830	≈ 9.7

Conclusions

In this presentation, on the basis of a few examples, we have shown that the proposed interval difference method give better enclosures of the exact solutions than those obtained by the well-known Nakao's method.

Although Nakao's method can be applied to a lot of elliptic boundary value problems, there are some inconveniences, from which the main one consists in non-applicability of the method to the well-known Poisson equation.

However, it should be added that the proposed interval method can be considered only as an experimental one. Since some quantities occurring in the method are adopted experimentally, a strictly mathematical proof that the obtained intervals contain exact solutions is rather impossible to receive.

But ...

Conclusions

If we assume that the constants M, P, Q and δ-extension are determined properly, then for the system of interval linear equations (14), from which we obtain our enclosures, we can quote the following theorem (see, e.g., [12, p. 89]):

If we can carry out all steps of a direct method for solving a finite system of linear algebraic equations $Ax = b$ in interval arithmetic (if no attempted division by an interval containing zero occurs, nor any overflow or underflow), the this system has a unique solution for every real matrix in A and every real vector in b, and the solution is contained in the resulting interval vector X.

---

[12] Moore, R. E, Kearfott, R. B., Cloud, M. J., Introduction to Interval Analysis, SIAM, Philadelphia (2009).

That's all
I wanted to present to you ...

... and thank you
for your attention