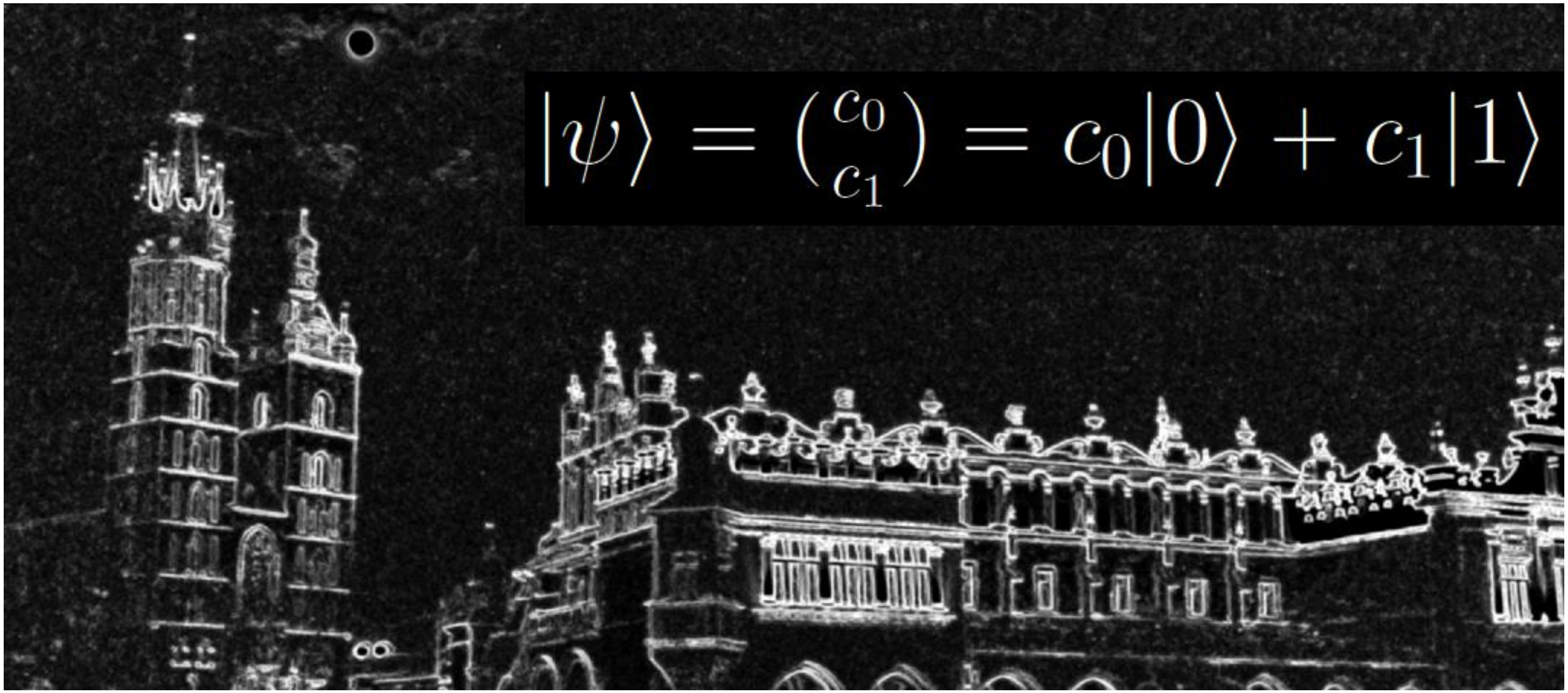
A personal Introduction to Quantum Information Theory



Karol Życzkowski (UJ / CFT PAN)

QIPLSIGML, Cracow, April 27 2018

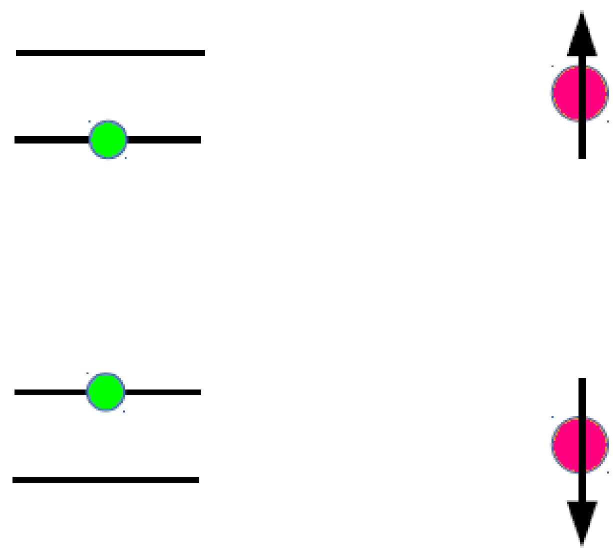
Otton Nikodym and Stefan Banach

talking in Cracow Planty Garden, summer 1916

Classical Information
= sequence of bits

bit (*binary unit*) = {0,1}

Logical bit ***physical realizations*** ***quantum notation***

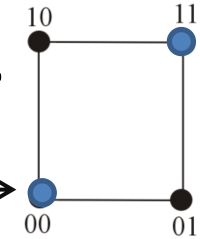


0 $|0\rangle$

1 $|1\rangle$

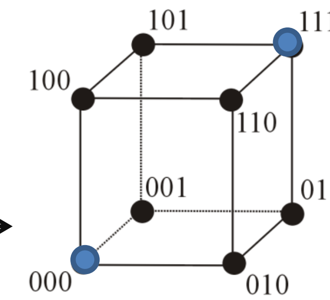
classical

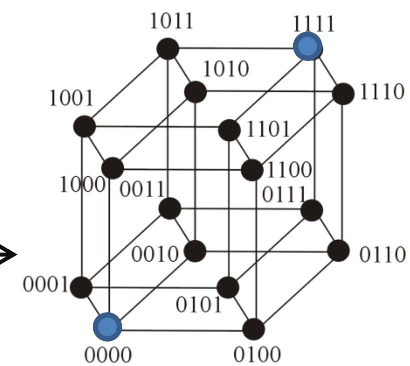
one *bit* → 2 states: {0, 1}



2 *bits* → *4* states: →

3 *bits* → *8* states: →





4 *bits* → *16* states: →

…

n *bits*

…

2^n states: {00⋯0, … ,11⋯1}

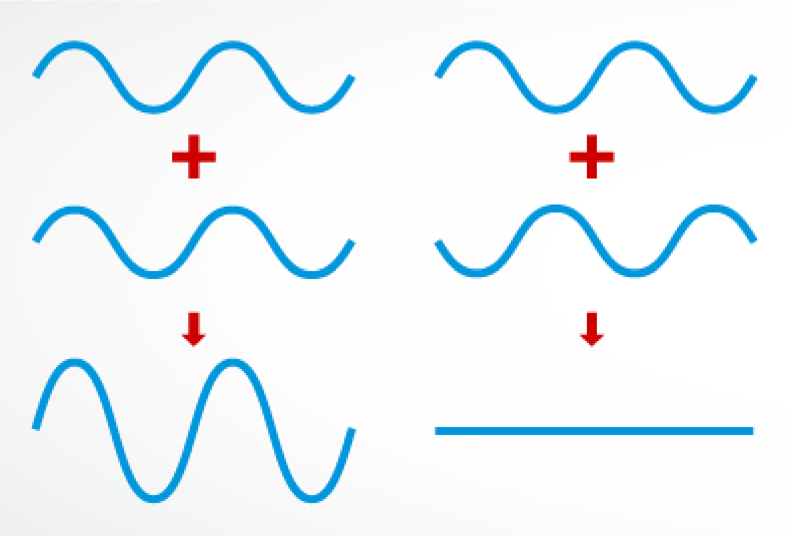
forming a hypercube =

Cartesian product: $\{0,1\}^{x\,n}$

Quantum physics is applicable at micro scale:

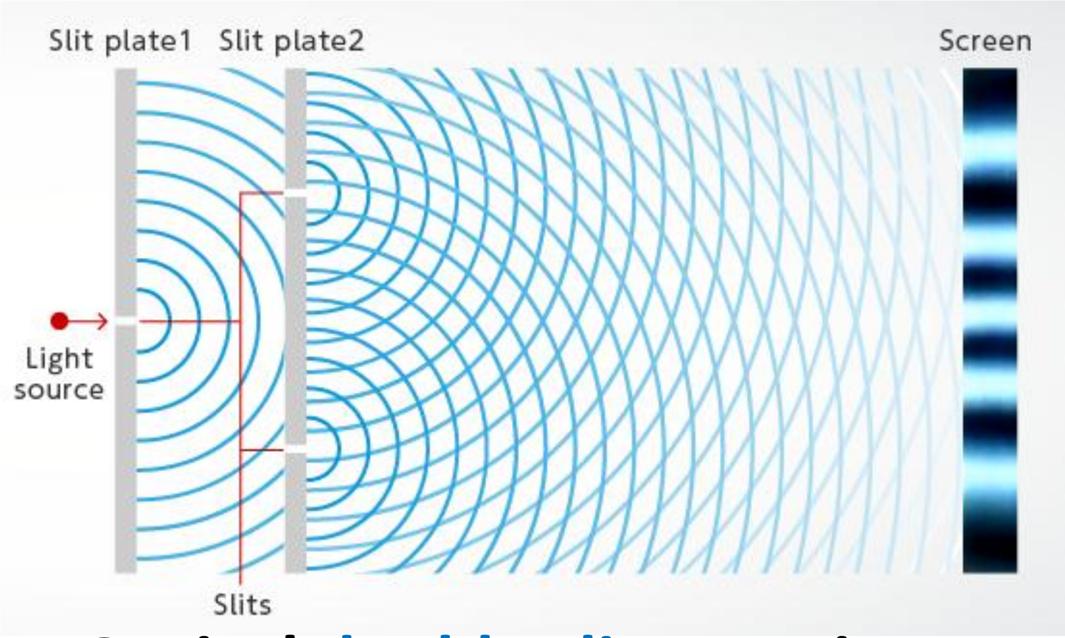
particle – wave duality explains

interference effects:



constructive   destructive

Young 's experiment (1807) with waves and two slits



Optical double slit experiment: a single photon displays interference fringes

Quantum State $|\psi\rangle$:

mathematical tool allowing one to compute probability of results measured

particle : wave function $|\psi(x)\rangle$

bit (*binary unit*) = {0,1}

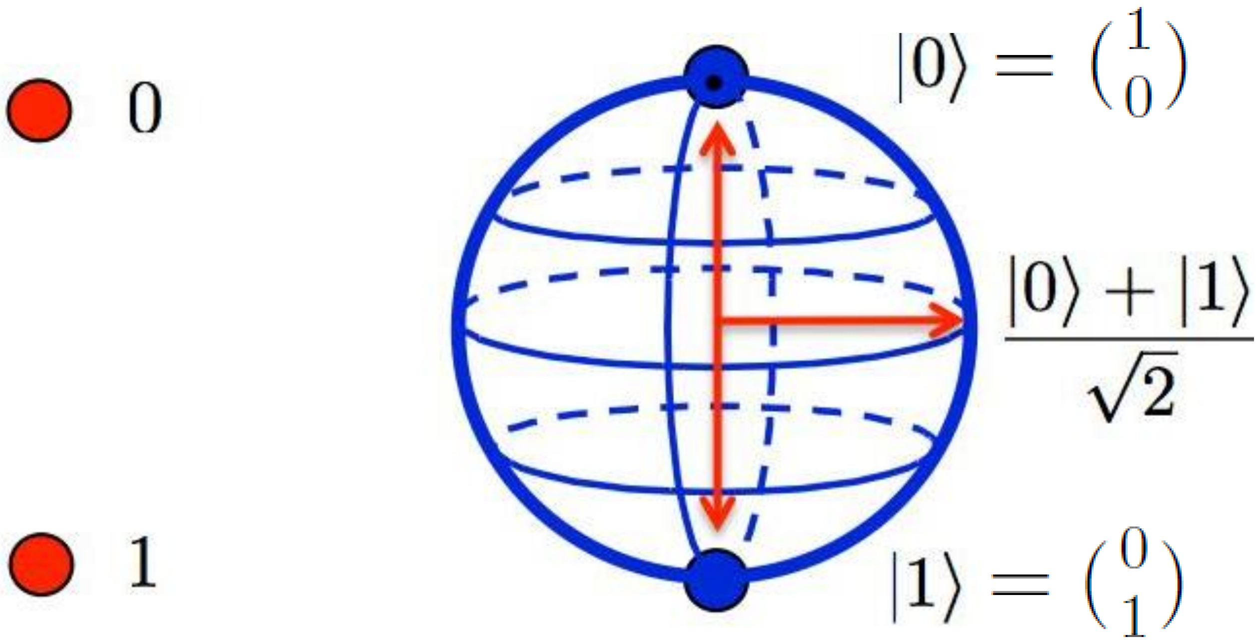
qubit = quantum bit =

a quantum system with two distinguishable states

complex vector $|\psi\rangle = \binom{c_0}{c_1} = c_0|0\rangle + c_1|1\rangle$

Components of the vector $|\psi\rangle = \binom{c_0}{c_1}$

determine probability to obtain a given outcome $p_0 = |c_0|^2, p_1 = |c_1|^2$



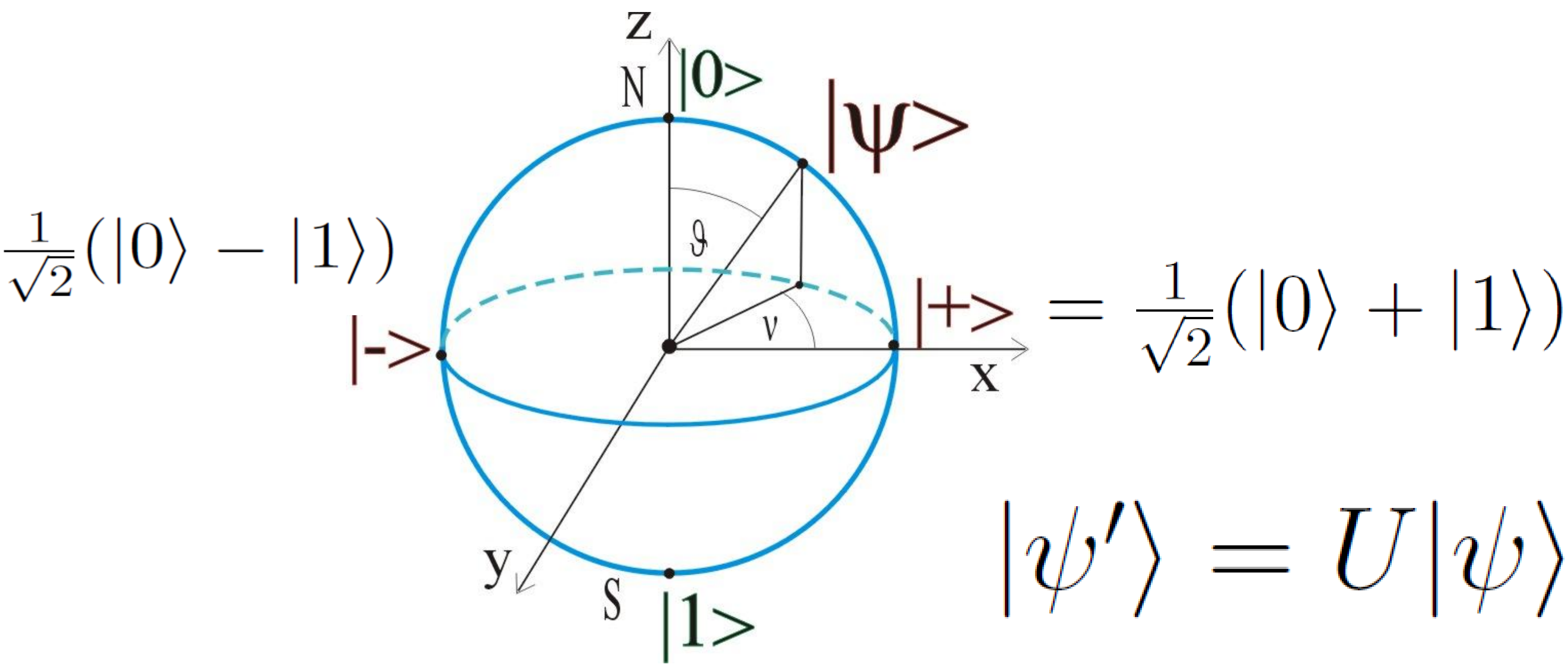
state of a qubit = point at the Bloch sphere

Normalization of the state :

$$p_0 + p_1 = |c_0|^2 + |c_1|^2 = 1$$

Classical states : |0>, |1>

Non - classical : |+>, |->, |ψ>



$$|\psi'\rangle = U|\psi\rangle$$

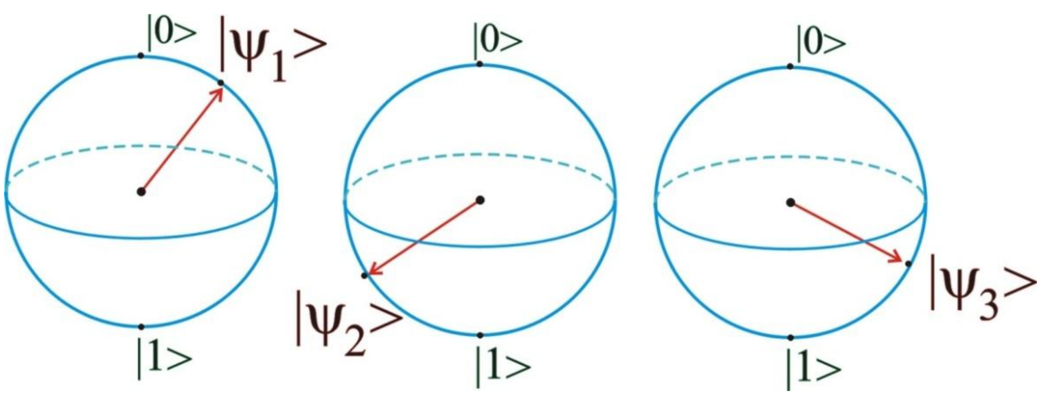
Quantum superposition – a pure state $|\psi\rangle$ differs from a classical mixture of quantum states (a point inside the Bloch ball)

Theory of classical information :

works with classical bits : {0,1}

Theory of quantum information :

relies of qubits : $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \ldots$



Advantages:

a) **Larger space of states allowed**

b) **Larger set of operations available**

Bipartite systems : {A,B}

a) **Separable** (product) state - no correlations !

$$|\psi_{\mathrm{sep}}\rangle = |\psi_A, \psi_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

b) **Entangled state** (not product) – shows correlations

$$|\psi_{\mathrm{ent}}\rangle \neq |\psi_A, \psi_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

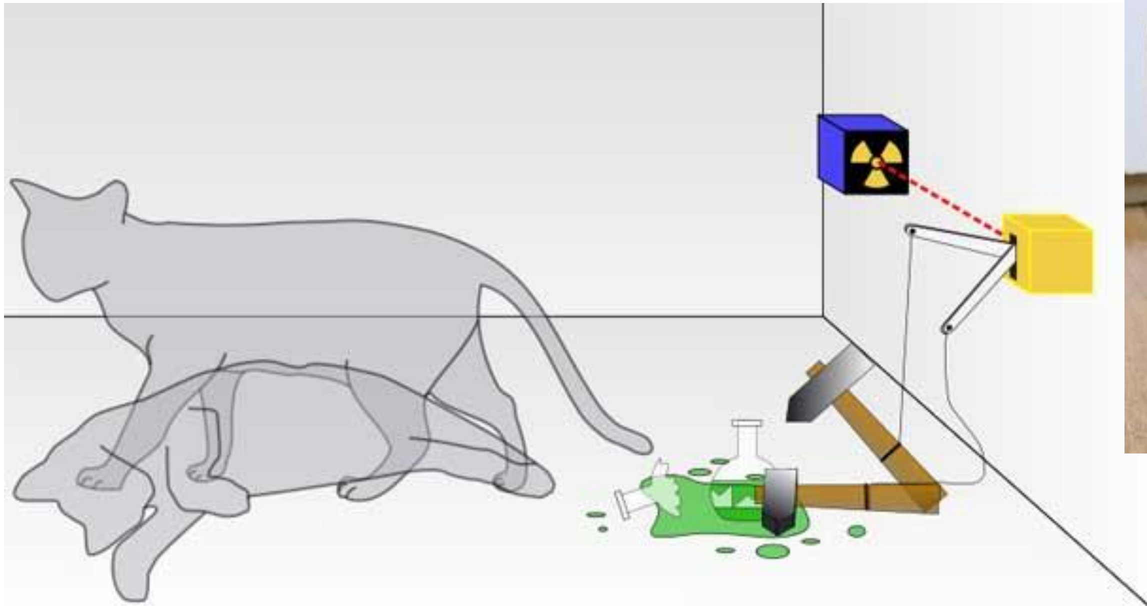
Superposition of two bipartite states

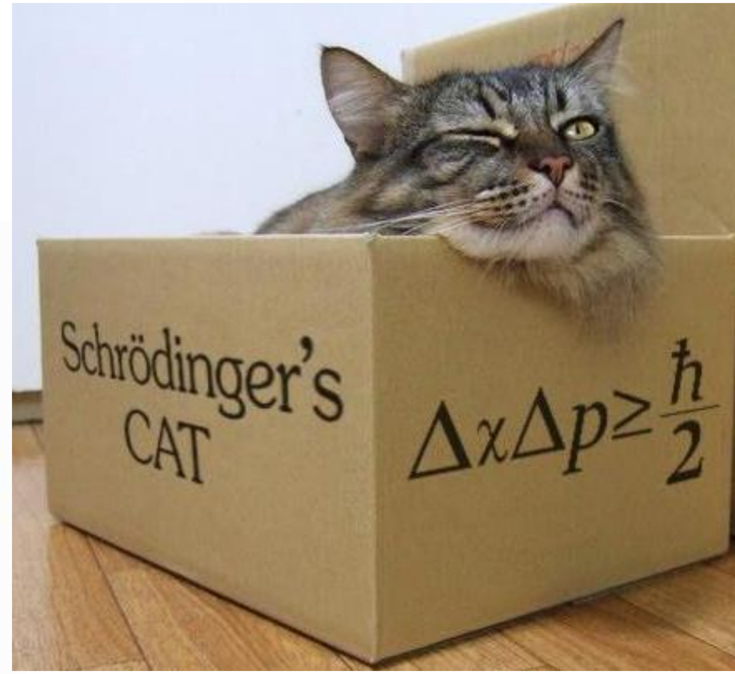
example: the **Bell state**

$$\boxed{|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}$$

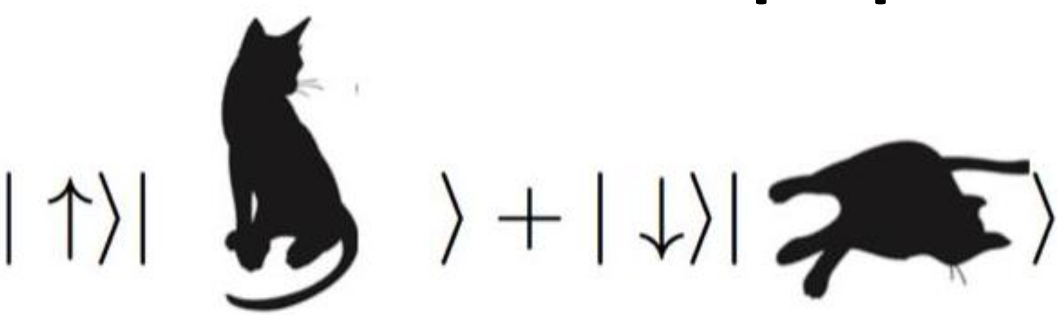
Entangled state reveals quantum correlations present due to previous interactions between subsystems

Schrödinger cat

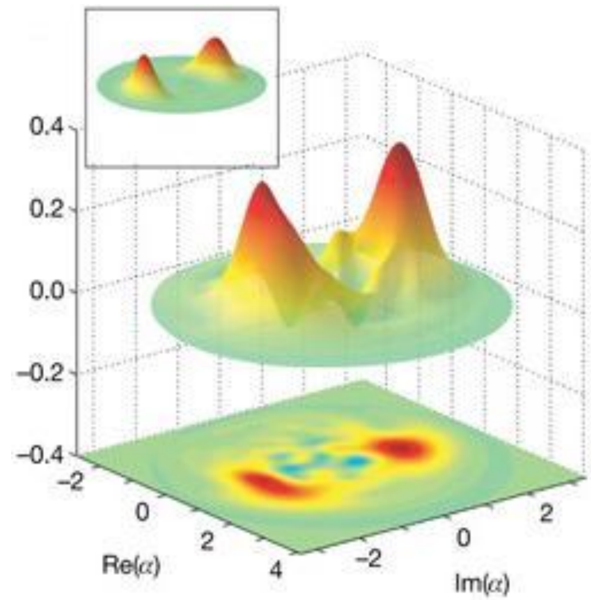




Quantum superposition :



entangled state

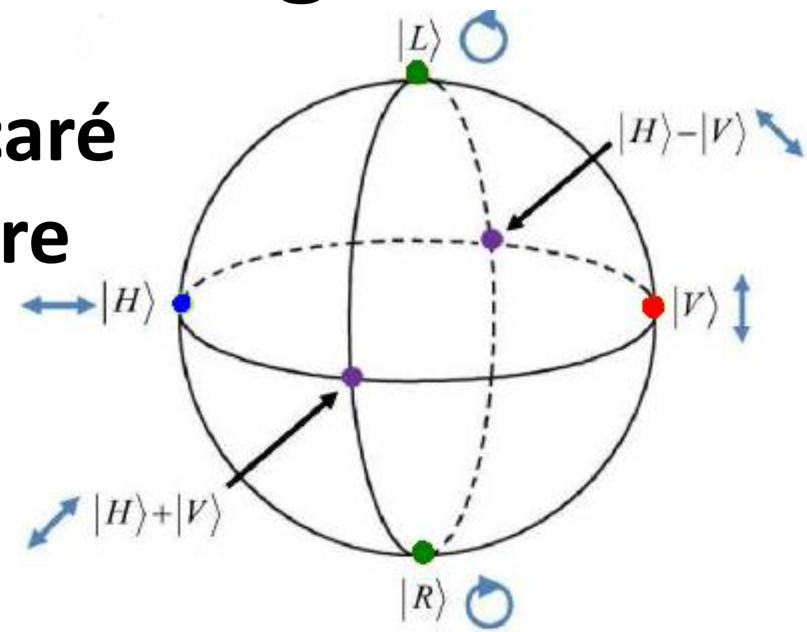


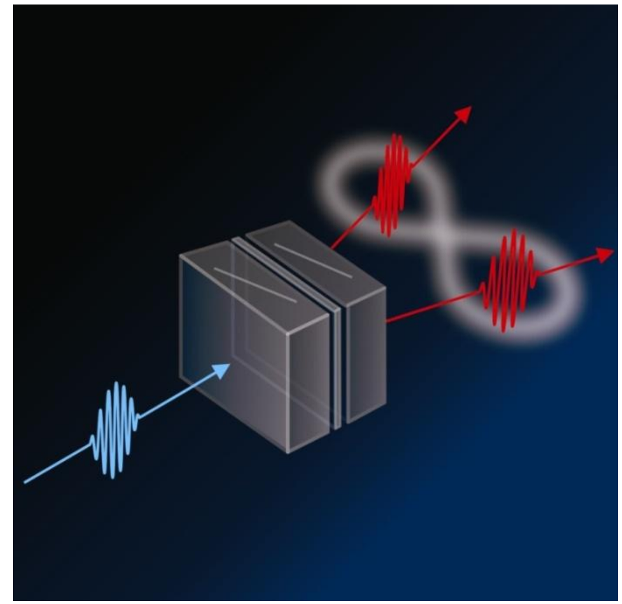
Haroche (et al.) 2008

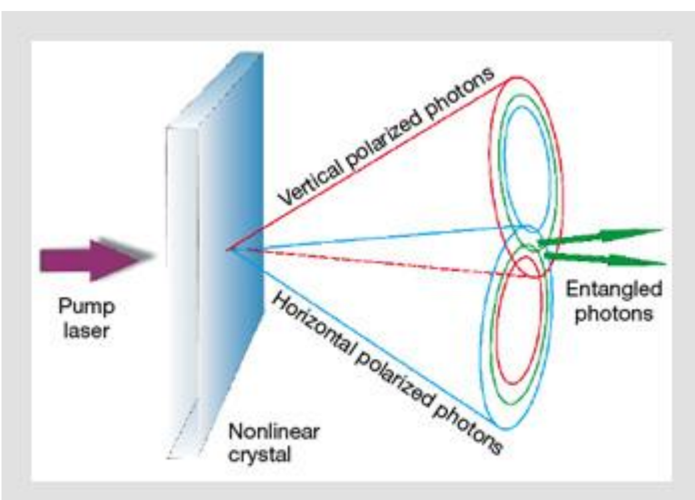
"About your cat, Mr. Schrödinger—I have good news and bad news."

Polarisation of light and entangled states of two photons

Poincaré sphere





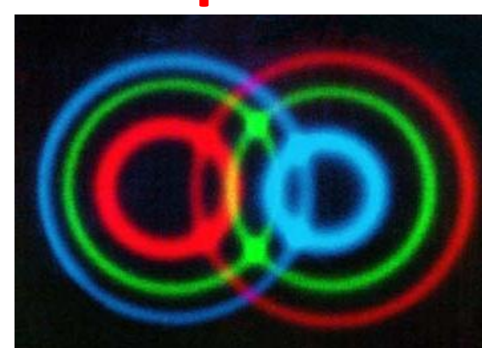


conversion: 1 blue photon

⟶ 2 red photons

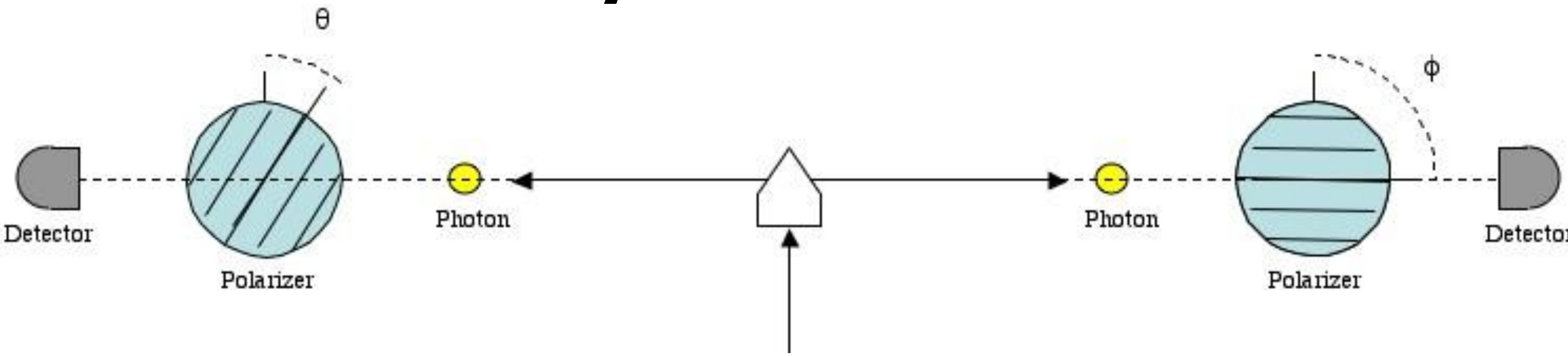
entangled state:

|HV> + |VH>



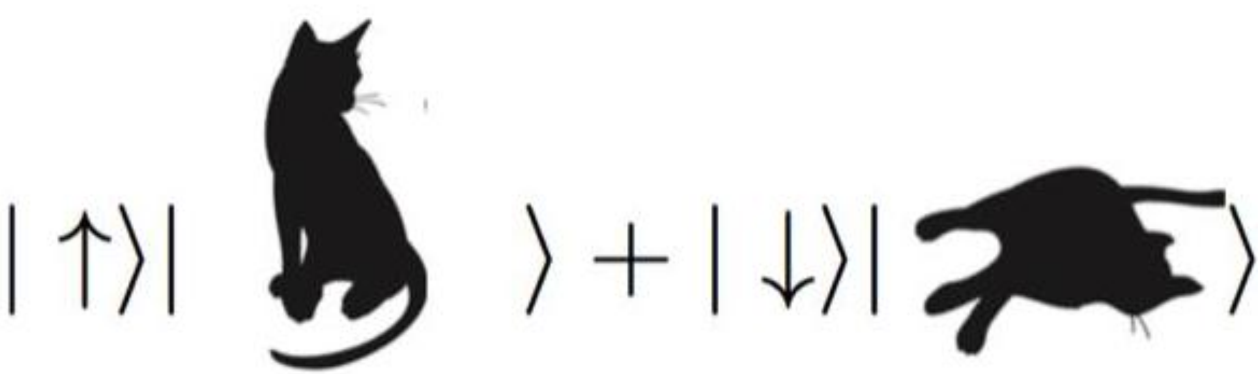
1. Quantum Entanglement:

correlations between results measured by two detectors

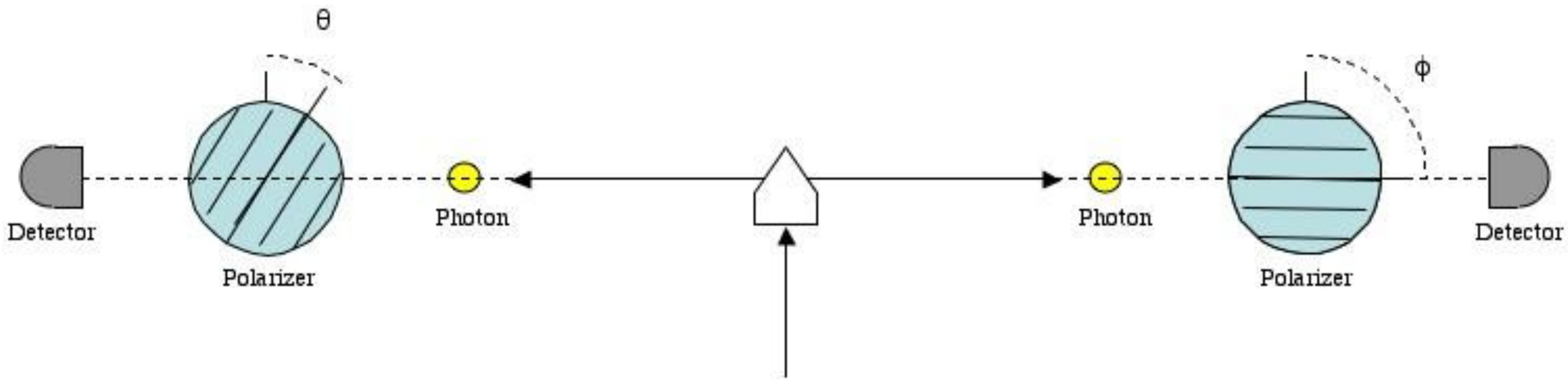


$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

2. Analogy to the Schrödinger cat



3. Existence of entangled states does not violate relativistic rules:



No superluminal information transmission !

4. Entangled states do exist, but they are fragile : they are destroyed by interaction with an environment and any quantum measurement !

Quantum information processing

makes use of :

superposition of states $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

and entanglement $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Applications:

* quantum teleportation
* quantum computing
* quantum cryptography

quantum metrology, simmulations

quantum games, finances,

quantum ... ...

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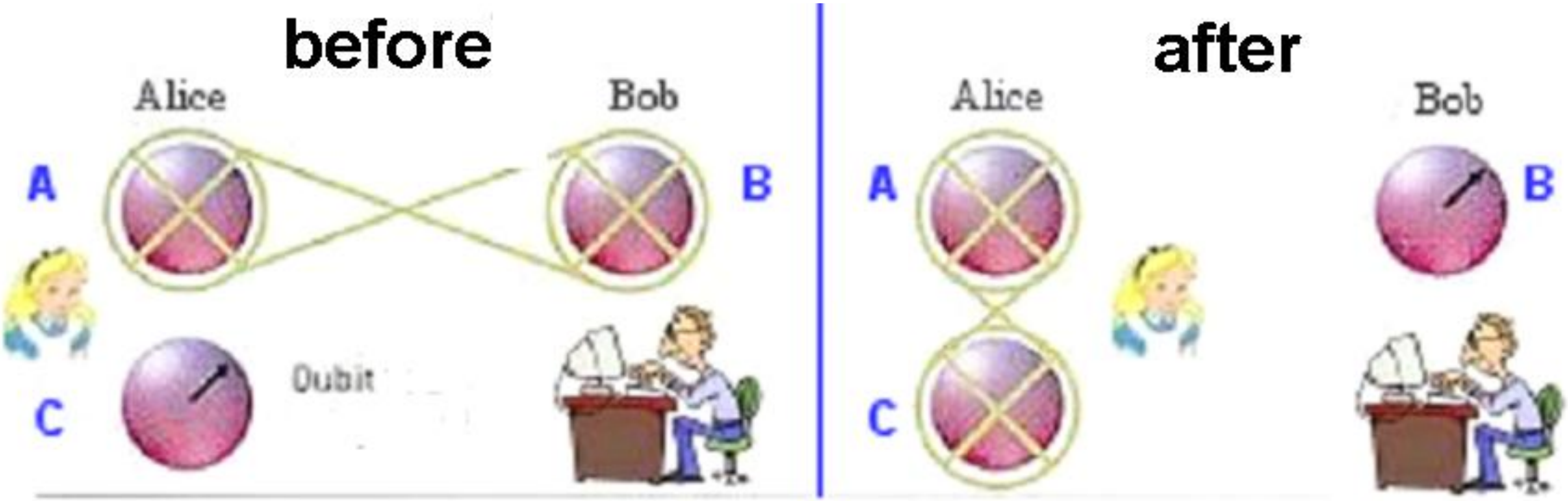
quantum metrology, simmulations

quantum games, finances

quantum machine learning !

… … …

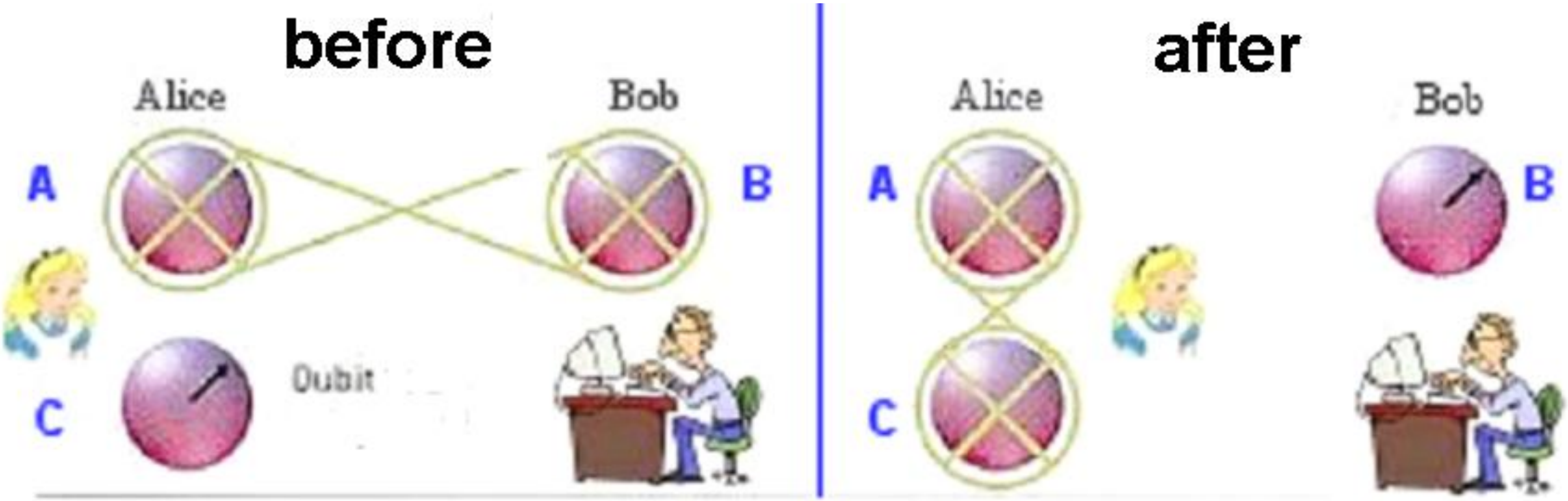
Scheme of quantum teleportation

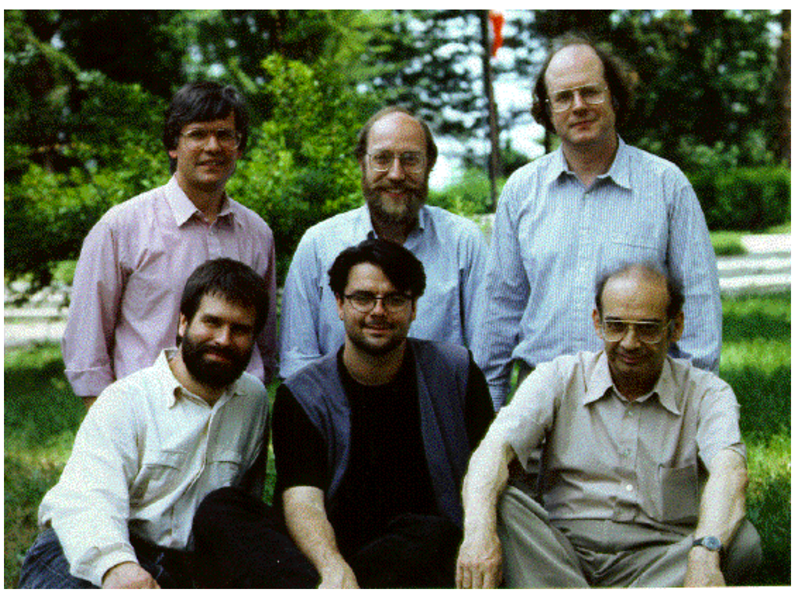


Making use of a Bell state
Alice sends her unknown
state C reconstructed by Bob

What is teleported ?

Scheme of quantum teleportation





Bennett et al. 1993

Making use of a Bell state
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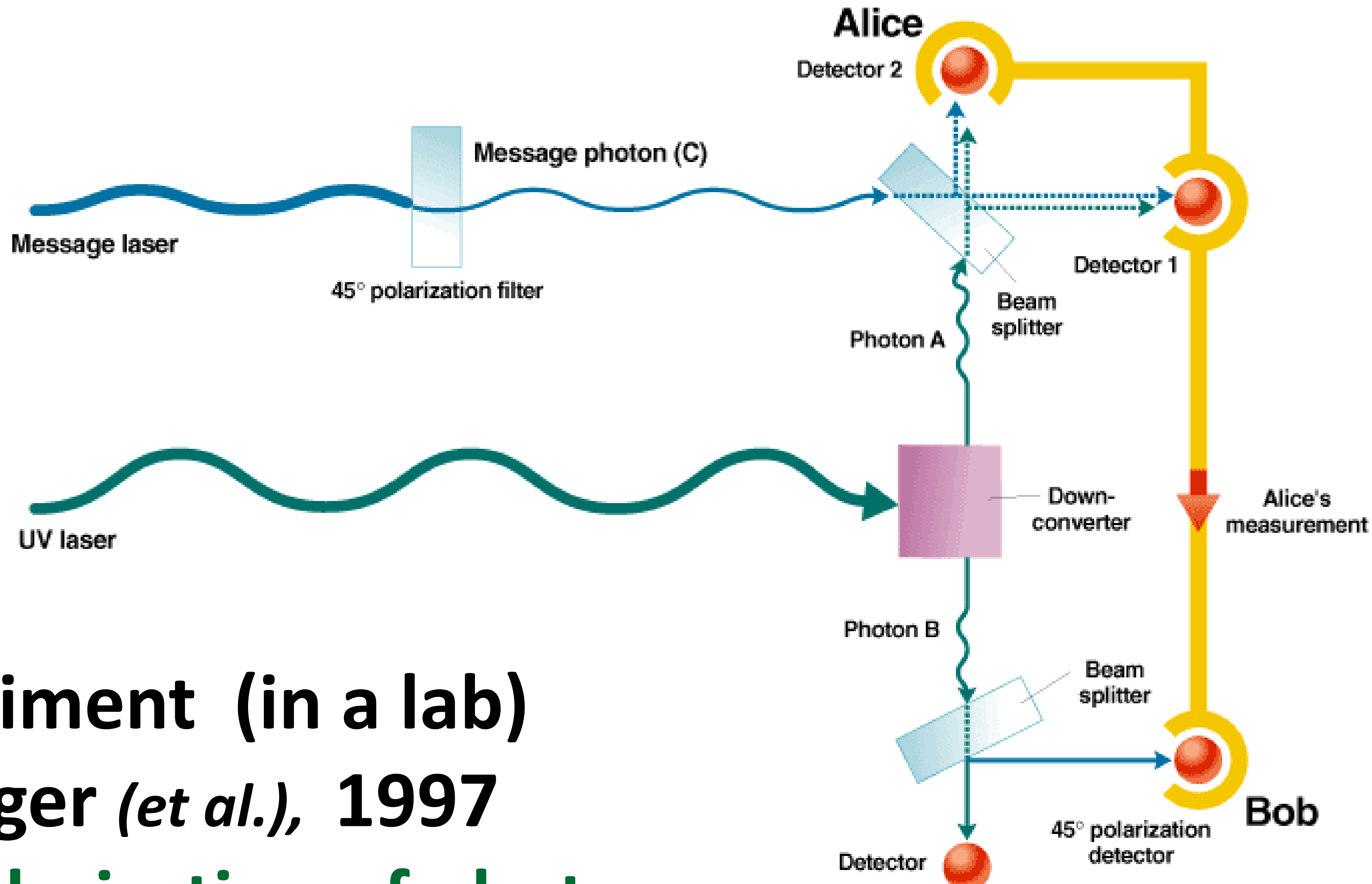
What is teleported ?

Quantum information !

quantum teleportation

theory: Bennett et. al. 1993

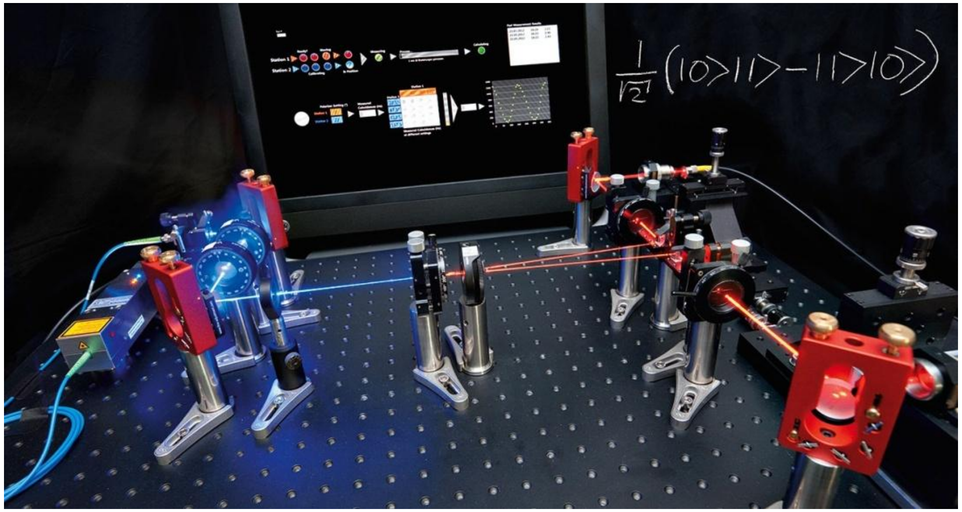


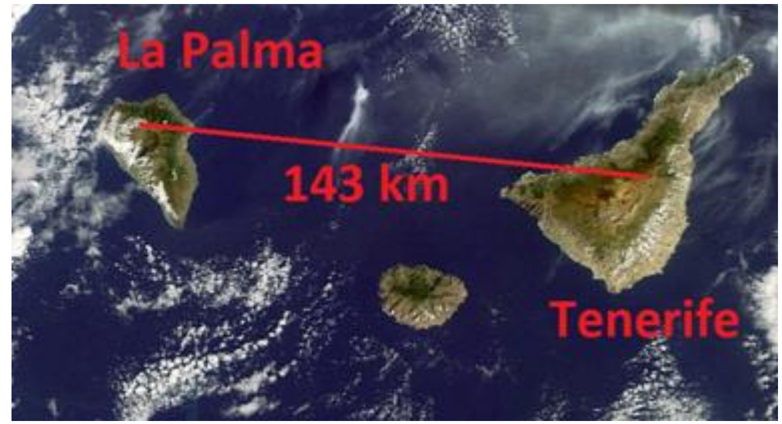
experiment (in a lab)

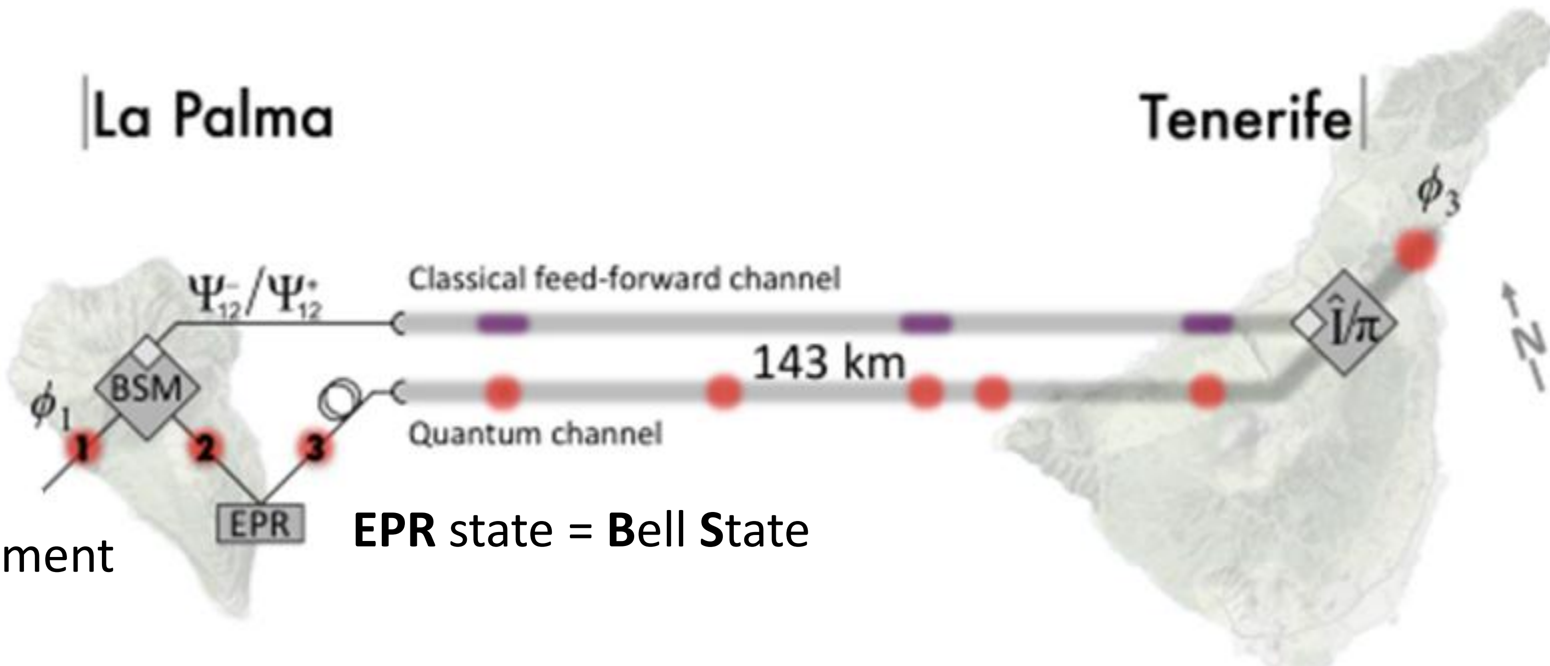
Zeilinger *(et al.),* 1997

polarization of photons

quantum teleportation at 143 km, Canary Islands
European Space Agency, 2012







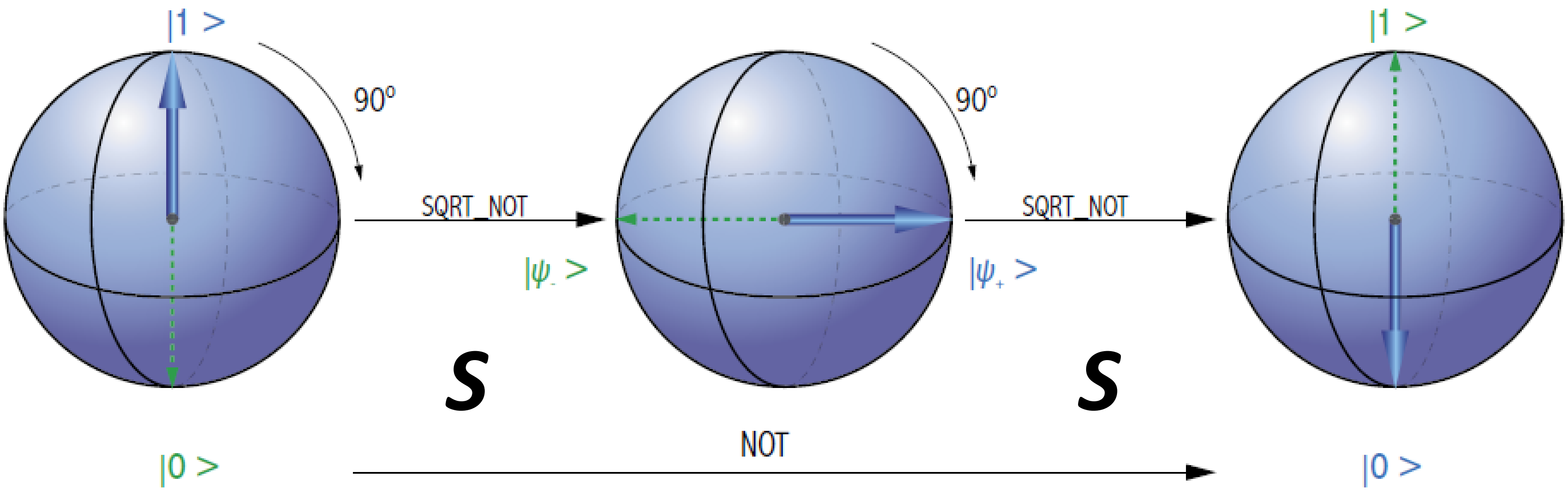
Bell
State
Measurement

EPR state = **B**ell **S**tate

Quantum dynamics:

$$|\psi'\rangle = U|\psi\rangle$$

example of a non-classical 1-qubit (local) gate :



S **= square root of *NOT* gate => *S*^{2} = *NOT***

Unitary evolution matrix *U* can be decomposed into elementary local gates and 2-qubit Control NOT gate

1-qubit **Hadamard** gate $$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

induces superposition, ***H*|0> = |+>** $= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

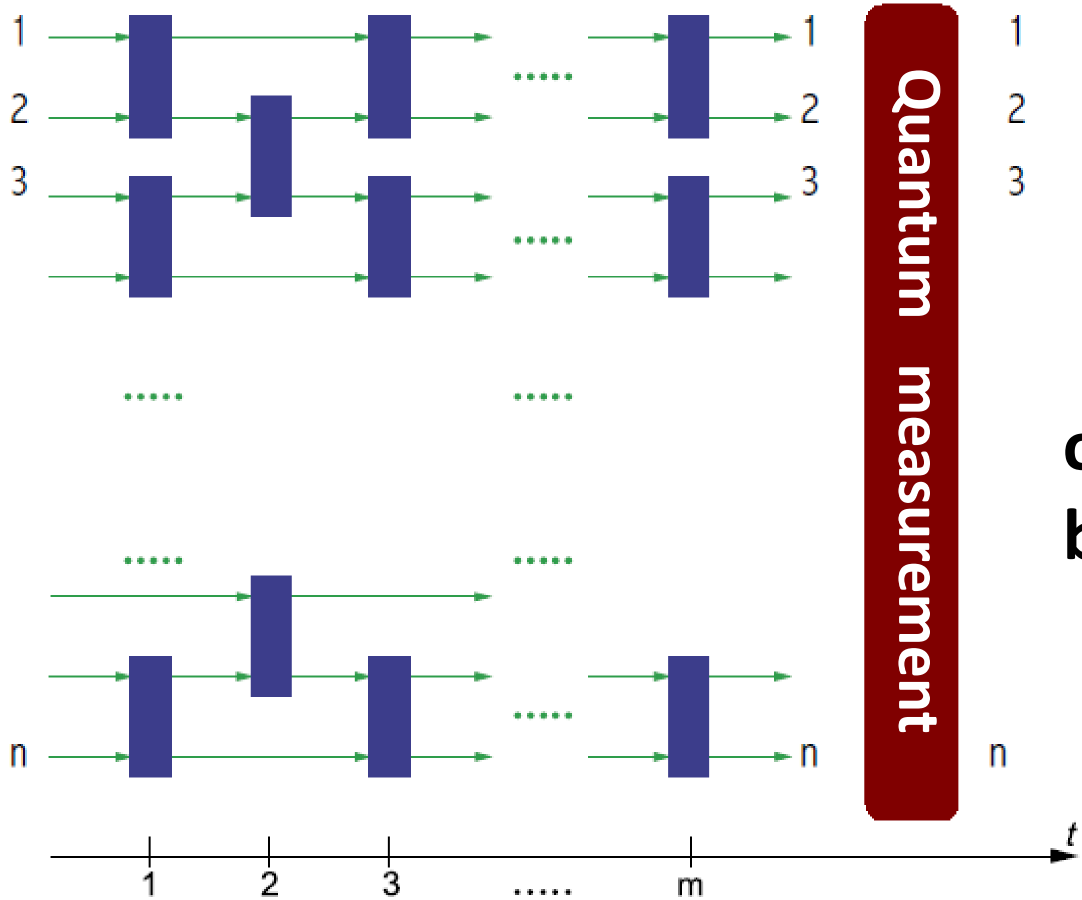
2-qubit **Control NOT** gate is non-local, and it $$U_{CNOT} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & \sigma_x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

creates entanglement,

$$U_{CNOT}|+,0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi^+\rangle$$

universal quantum computation scheme

n
input
qubits



n
output
bits

unitary quantum gates measurement

Measurement brings inherent randomness:

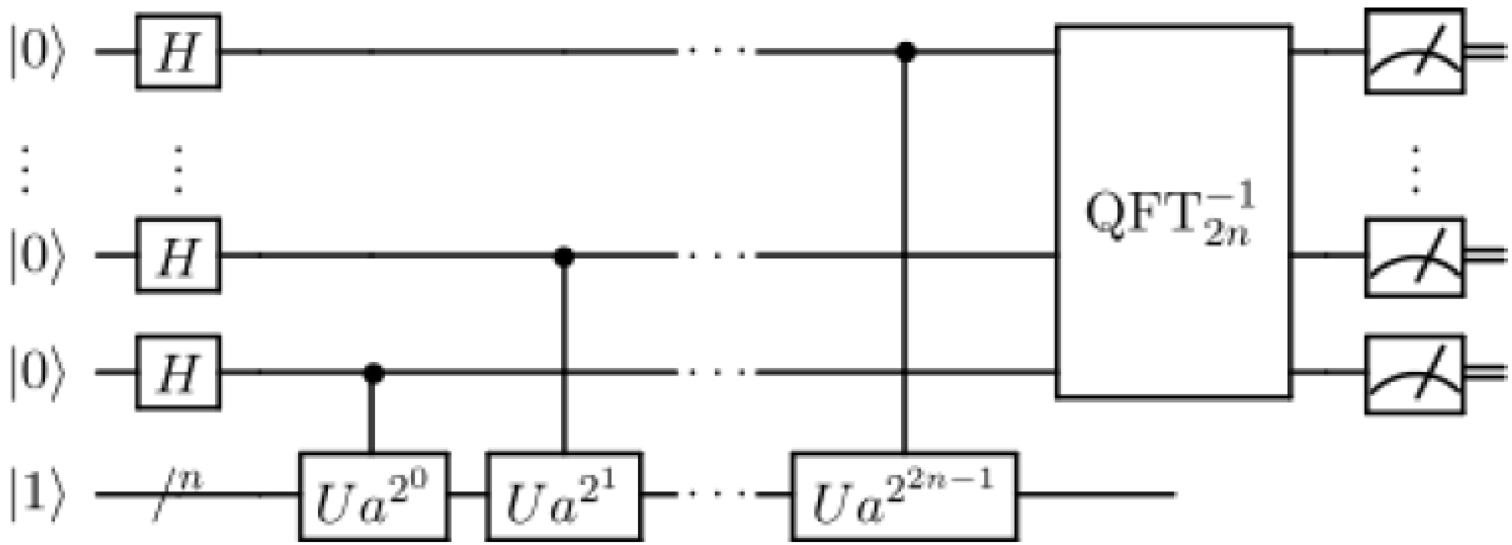
for state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ probability $p_0 = |c_0|^2$

Example: Shor's algorithm (1994)

Factorization of a number *M* is as difficult as finding the period *T* of a periodic function, $f(x)= f(x+T)$

Quantum Fourier transform (QFT)

a unitary operation acting on *n* qubits with $M < 2^n$



Complexity measured by number of operations :

classical ~ $\exp(M^{1/3})$ quantum ~ $M^3 \log M$

quantum cryptography: key idea

a) If we **measure** a quantum state, we **alter** it

b) **Observation** of a quantum system **modifies** it

c) Any **action** of an evesdropper can be detected

quantum key generation (*truly random digits*)
quantum key distribution (*ensures security*)

quantum cryptography: key idea

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quantum key generation (*truly random digits*)
quantum key distribution (*ensures security*)

Commercial products, e.g.

ID Quantique ***(Geneve)***

Current trends in theory of quantum information:

seach for new

- algorithms for quantum computing
- ideas to demonstrate quantum supremacy
- techniques to process quantum information
- original applications

but also preparatory work

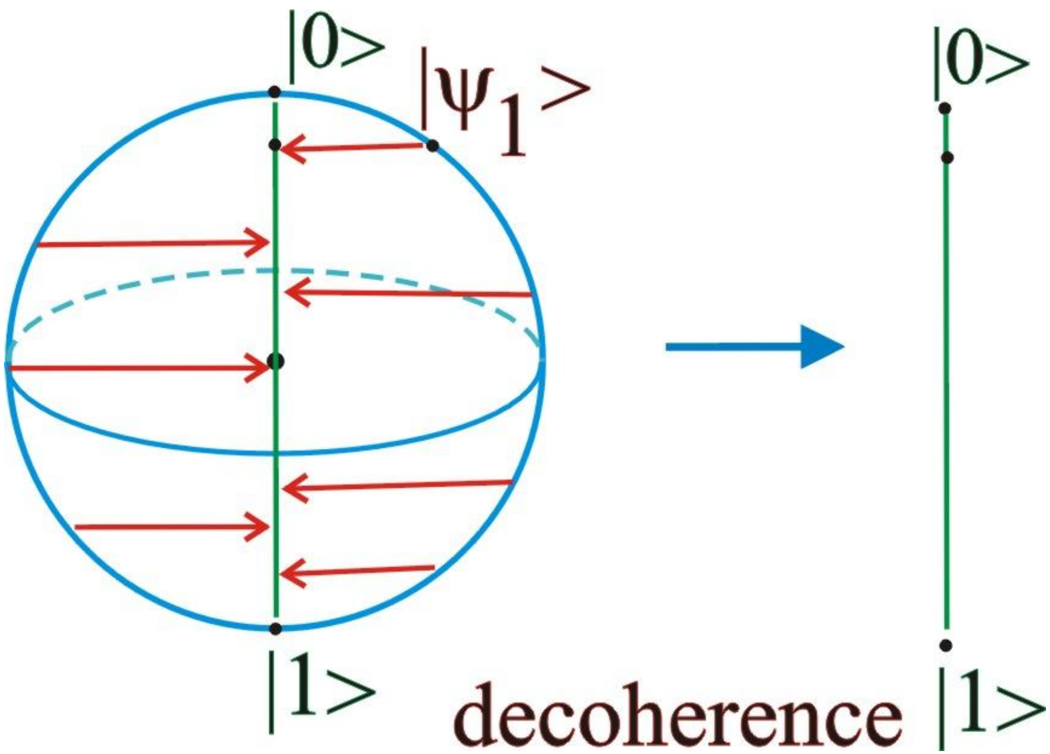
- studies on quantum states and operations
- investigations of correlations and entanglement
- foundations of quantum theory
- mathematics motivated by quantum information
- … … …

Theory of quantum information:

some features (and problems):

a) results are probabilistic - need for repetitions

b) quantum superposition gets spoiled due to interaction of a qubit with an environment



decoherence:

a pure state |ψ>

is transformed

into a classical

mixed state

To describe interaction with environment one applies formalism of **density matrices** – so called **mixed states**

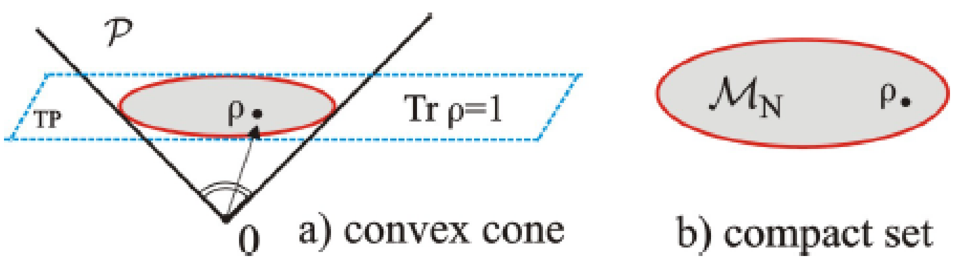
Set $\mathcal{M}_N$ of all mixed states of size N

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

example: $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - Bloch ball with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and **convex**: $\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i|$, where $a_i \geq 0$ and $\sum_i a_i = 1$.

It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.



What the set of all $N = 3$ mixed states looks like?

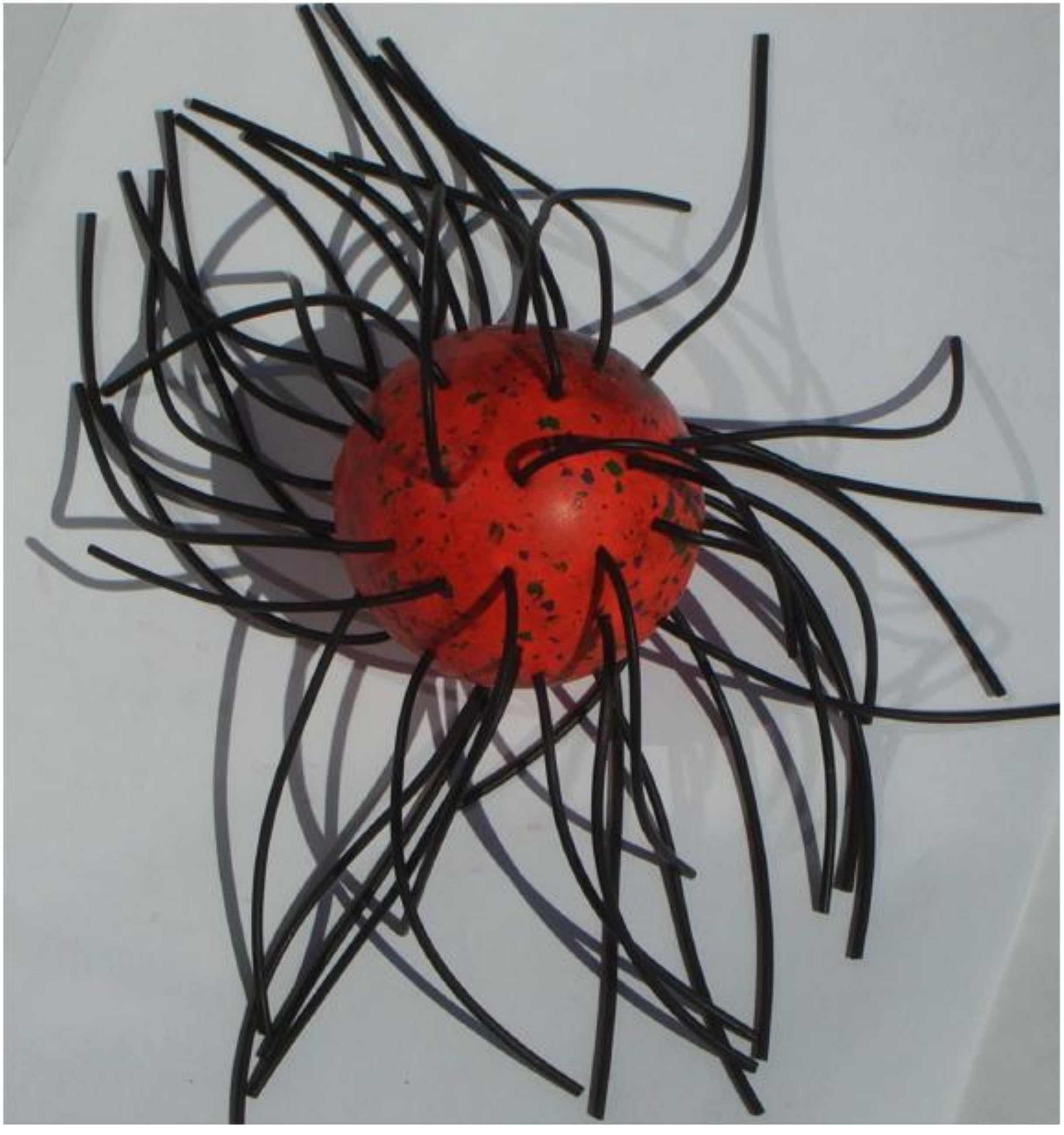
An 8–**dimensional** **convex** set with only 4–dimensional subset of pure (**extremal**) states, which belong to its 7–dim boundary

The set $\mathcal{M}_N$ of quantum mixed states:

What it looks like for **(for $N \geq 3$)**

?

An apophatic approach :

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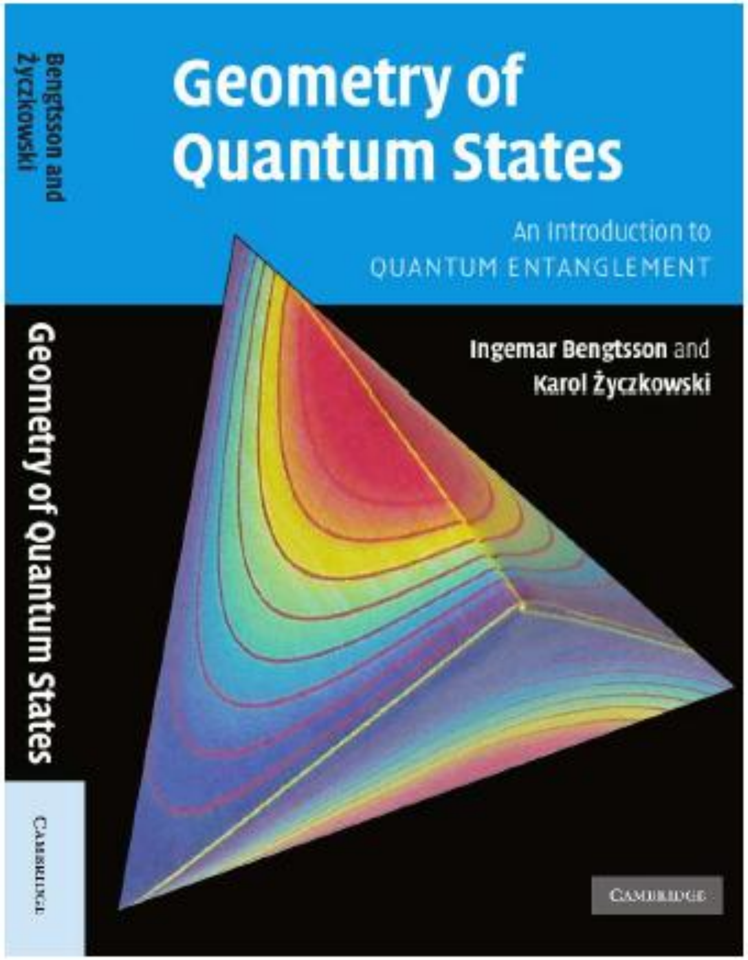
Tea Bags
APPEL THEE

We wish to improve our understanding

of the geometry of the **set of quantum states**

Book published by Cambridge University Press in **2006**,



by
Ingemar Bengtsson (Stockholm)
and K.Ż.

Entanglement of mixed quantum states

Mixed states of a bi-partite system, (A, B)

- **separable mixed states:** $\rho_{\rm sep} = \sum_j p_j \rho_j^A \otimes \rho_j^B \quad (**)$
- **entangled mixed states:** all states not of the above product form.

How to find,
whether a given density matrix ρ can be written in the form (**)
and is **separable ?**

The **separability problem** is solved only for the simplest cases of 2×2 and 2×3 problems...

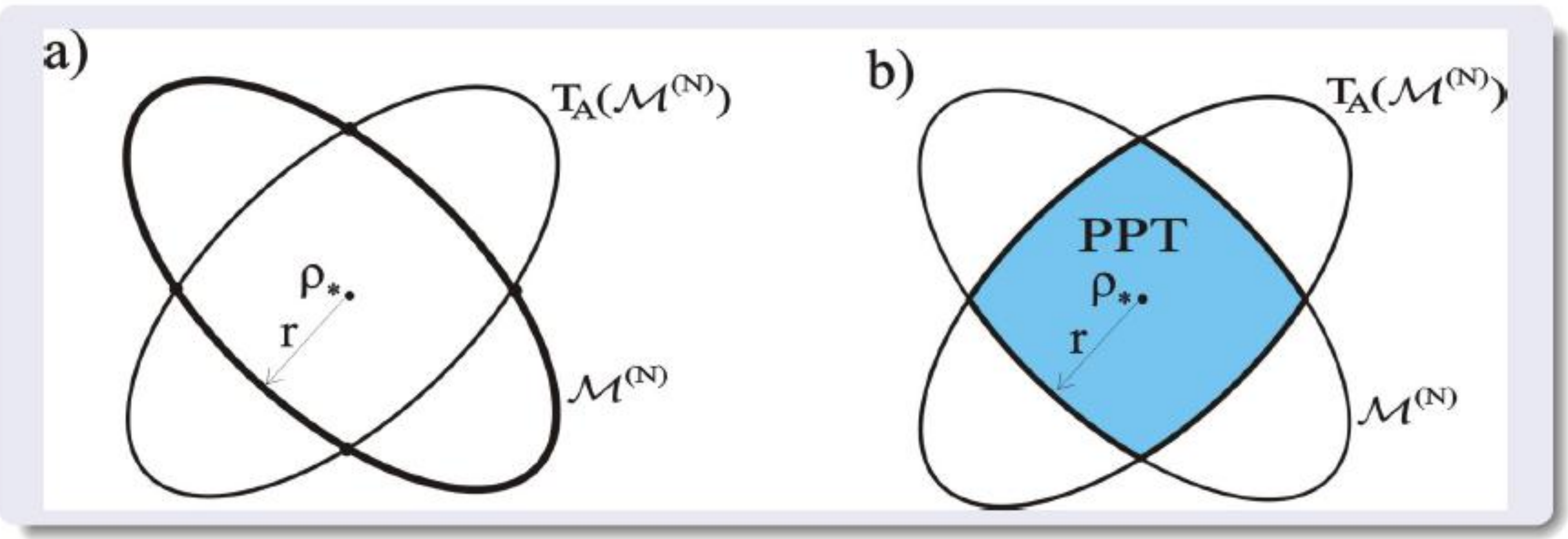
How to detect quantum entanglement?

Positive partial transpose: Two–qubit mixed states

Peres – Horodeccy criterion (1996):

$$(\mathbb{I} \otimes T)\rho = \rho^{T_2} \geq 0 \;\Leftrightarrow\; \rho \text{ is separable.}$$

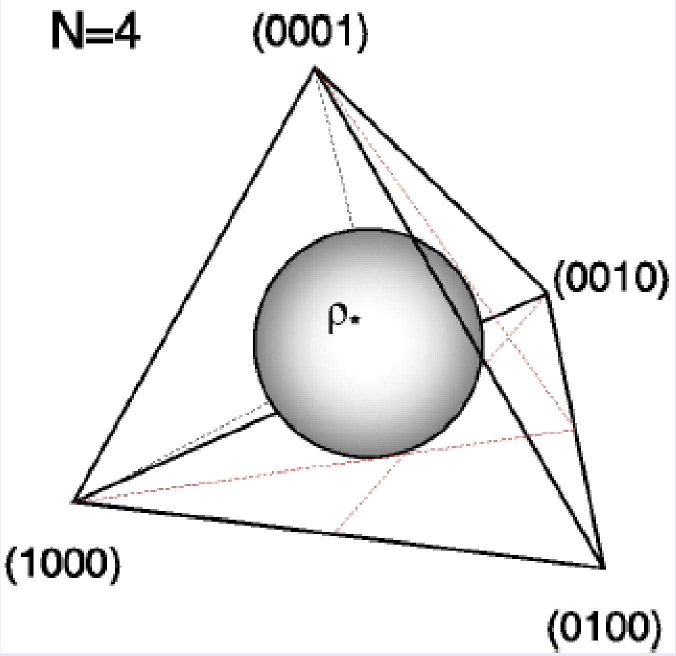
The set of separable states of two–qubit system arises as an intersection of $\mathcal{M}^{(4)}$ and its mirror image with respect to partial transposition $T_A(\mathcal{M}^{(4)})$.



How the set of separable / entangled states looks like?

Two–qubit mixed states

The maximal ball inscribed into $\mathcal{M}^{(4)}$ of radius $r_4 = 1/\sqrt{12}$ centred at $\rho_* = \mathbb{1}/4$ is separable !



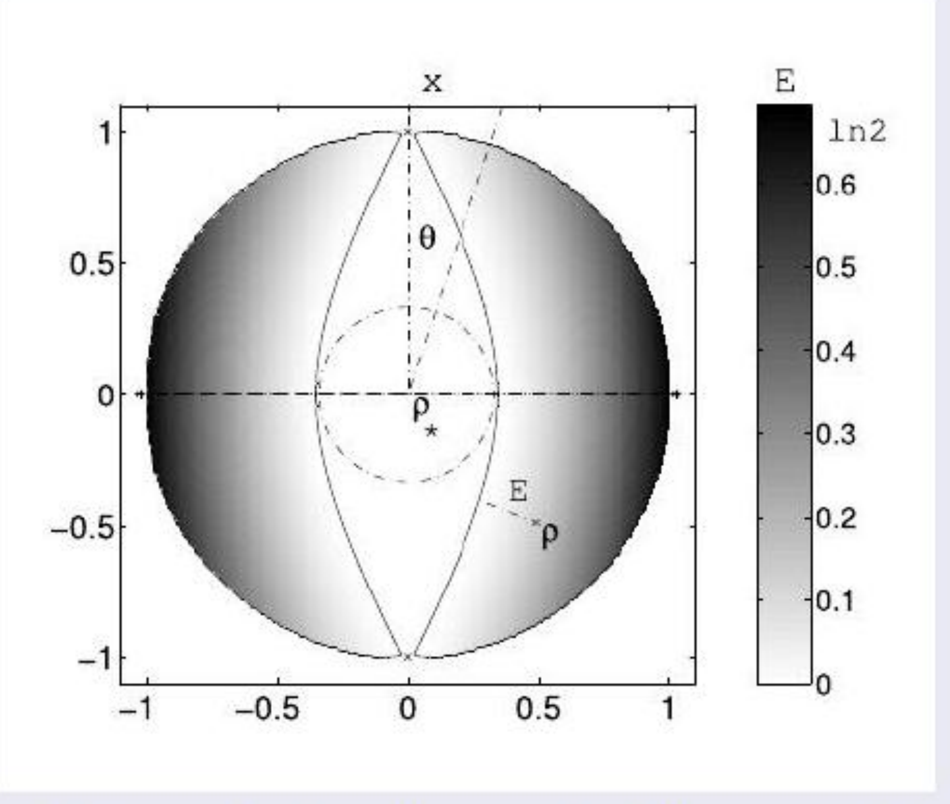
thetrahedron of eigenvalues

K. Ż, P. Horodecki, M. Lewenstein, A. Sanpera, 1998

How to quantify entanglement?

Two–qubit mixed states

Degree of entanglement: a **distance** to the closest **separable state**



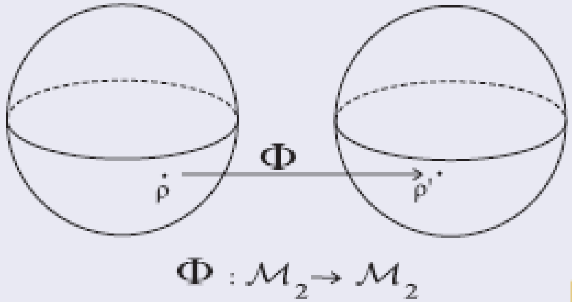
(E = **Entanglement** of formation)

K. Ż, M. Kuś, 2001

How to describe non-unitary quantum dynamics ?

Quantum maps

Quantum operation: linear, completely positive trace preserving map



positivity: $\Phi(\rho) \geq 0, \quad \forall \rho \in \mathcal{M}_N$

complete positivity: $[\Phi \otimes \mathbb{1}_K](\sigma) \geq 0, \quad \forall \sigma \in \mathcal{M}_{KN}$ and $K = 2, 3, \ldots$

Enviromental form (interacting quantum system !)

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes \omega_E)\,U^\dagger]\ .$$

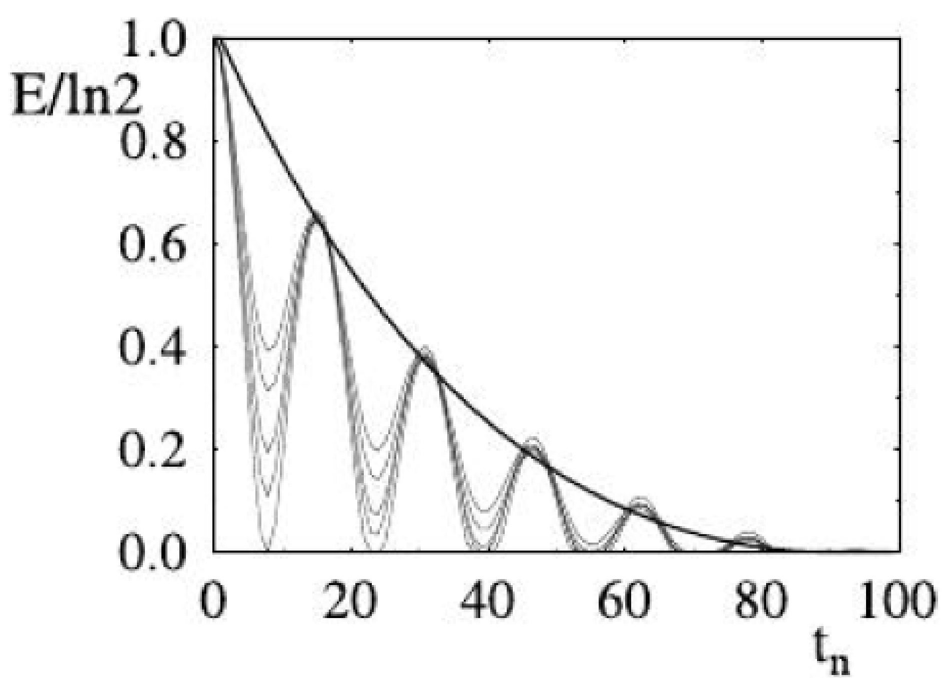
where ω_E is an initial state of the environment while $UU^\dagger = \mathbb{1}$.

How entanglement evolves in time ?

Dynamics of entanglement

Entanglement of formation E as a function of time t_n

for some initially pure states of a two–qubit system.



revivals of entanglement

Dynamics of entanglement

Entanglement of formation E as a function of time t_n

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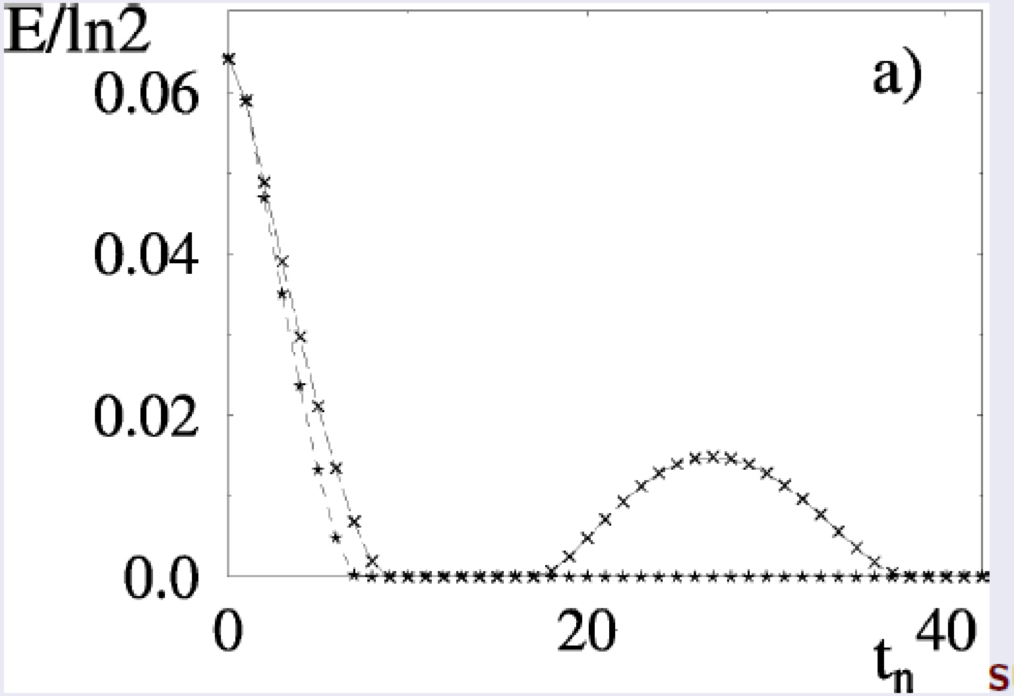


sudden death of entanglement

K. Ż, P. Horodecki, M. Horodecki, R. Horodecki, Phys. Rev. **A 2001**

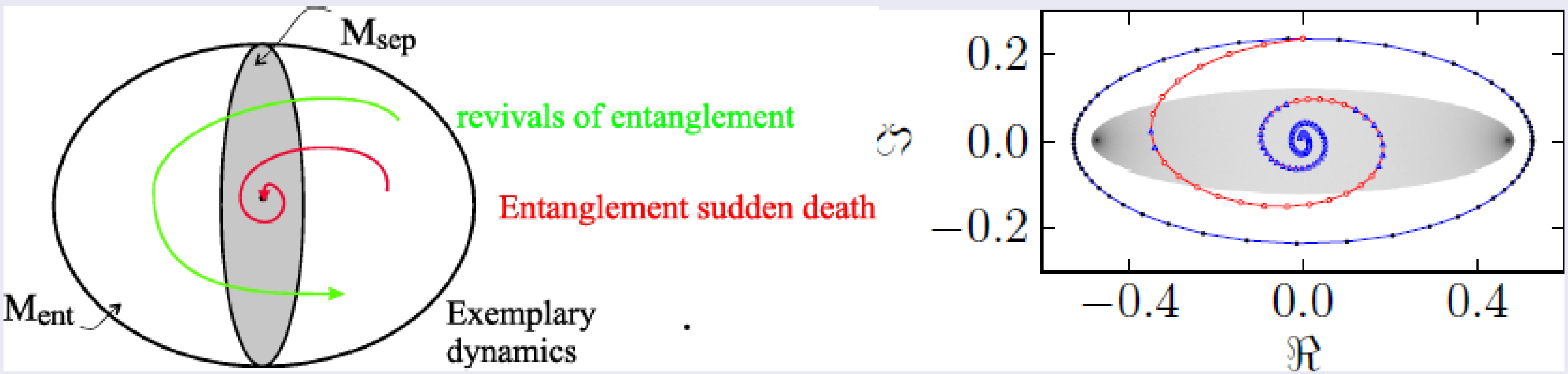
the name **sudden death** coined by **Yau and Eberly**,

who reported this effect in 2003.

How entanglement evolves in time ?

Trajectories of quantum dynamics on the complex plane

$$z(t) = \langle \psi(t) | A | \psi(t) \rangle$$



a) sketch of the problem; b) data for 2×2 system
with initial separable pure state $|\psi(0)\rangle$
and suitably chosen (non–Hermitian !) operator A of size $N = 4$
visualize possible behaviour of quantum entanglement...

How entanglement evolves in time ?

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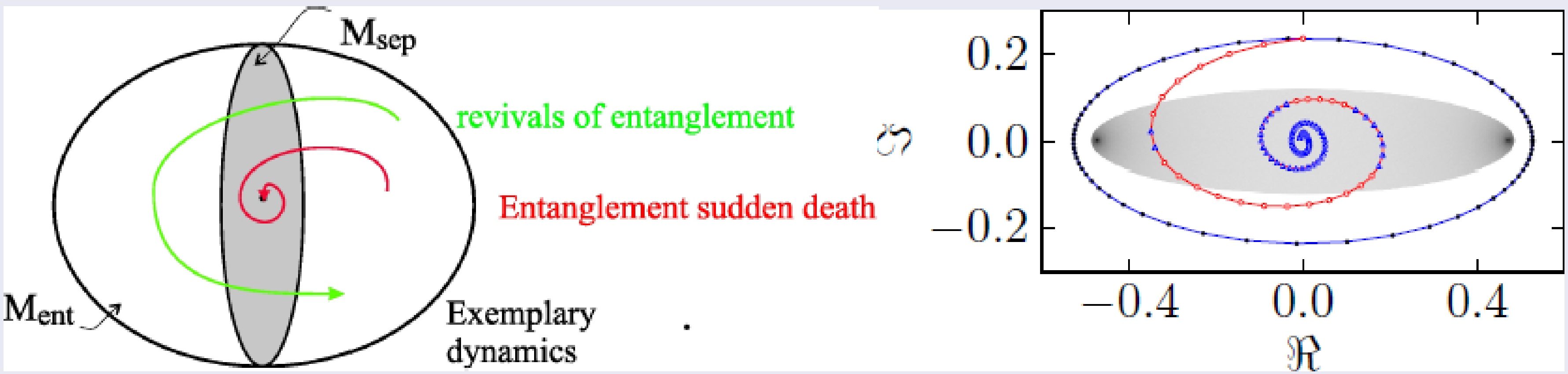

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visualize possible behaviour of quantum entanglement...

Z. Puchała, J.A. Miszczak, P. Gawron, C. Dunkl J. Holbrook, K.Ż. (2012)

GHZ state of three qubits:

(Greenberger–Horne–Zeilinger)

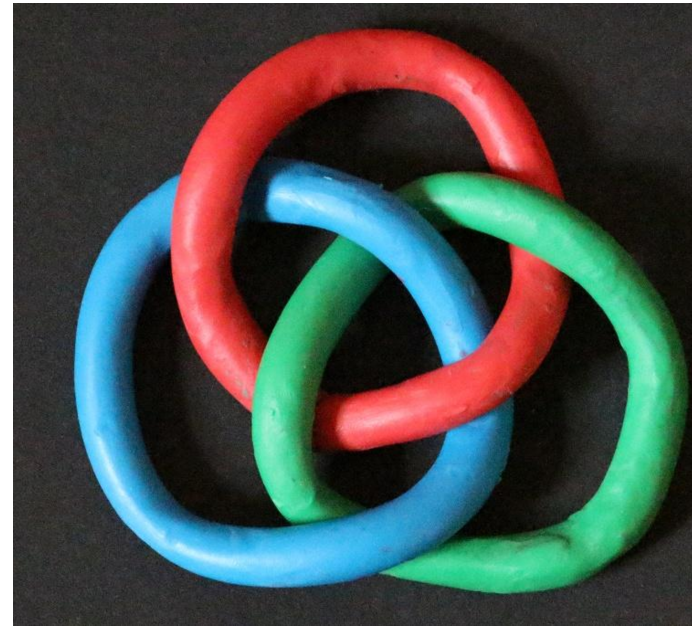
$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$$

enjoys properties analogous to Borromean Rings :

as a single ring is broken, two other are not connected any more.

GHZ state of three qubits:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$$



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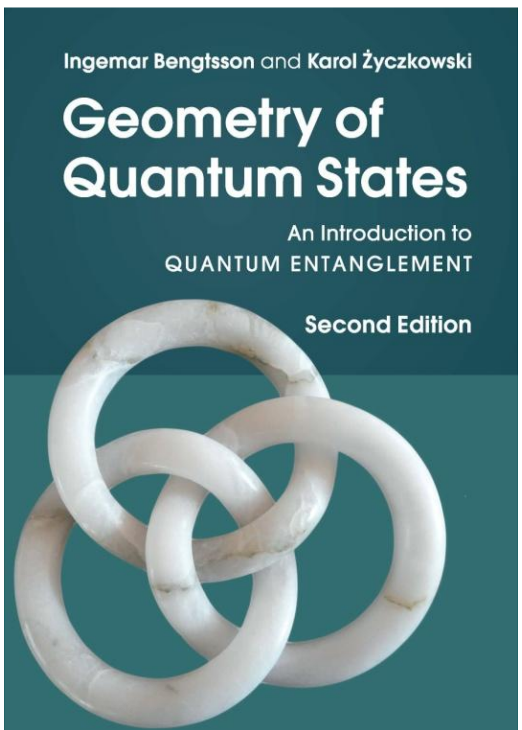

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Geometry of Quantum States

Ingemar Bengtsson, K.Ż.

II second extended edition

Cambridge University Press, **2017**

Which state is the most entangled?

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states **not** of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Schmidt decomposition & Entanglement measures

Any pure state from $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle = \sum_{ij} G_{ij} |i\rangle \otimes |j\rangle = \sum_i \sqrt{\lambda_i} |i'\rangle \otimes |i''\rangle, \text{ where } |\psi|^2 = \mathrm{Tr}\, GG^\dagger = 1.$$

The partial trace, $\sigma = \mathrm{Tr}_B |\psi\rangle\langle\psi| = GG^\dagger$, has spectrum given by the **Schmidt vector** $\{\lambda_i\}$ = squared **singular values** of G.

Entanglement entropy of $|\psi\rangle$ is equal to **von Neumann entropy** of the reduced state σ

$$E(|\psi\rangle) := -\mathrm{Tr}\, \sigma \ln \sigma = S(\lambda).$$

Multipartite pure quantum states

are determined by a **tensor**:
e.g. $|\Psi_{ABC}\rangle = \sum_{i,j,k} T_{i,j,k} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C$.

Mathematical problem: in general for a **tensor** there is no (unique) **Singular Value Decomposition** and it is not simple to find the **tensor rank** or **tensor norms** (nuclear, spectral).

Open question: Which state of N subsystems with d–levels each is the **most entangled** ?

example for three qubits, $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C = \mathcal{H}_2^{\otimes 3}$

GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$ has a similar property:
all three one-partite reductions are **maximally mixed**
$\rho_A = Tr_{BC}|GHZ\rangle\langle GHZ| = \mathbb{1}_2 = \rho_B = Tr_{AC}|GHZ\rangle\langle GHZ|$.

(what is **not** the case e.g. for $|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$)

Absolutely maximally entangled (AME) states

Higher dimensions: AME(4,3) state of four qutrits

From OA(9,4,3,2) we get a 2–**uniform** state of **4 qutrits**:

$$
\begin{aligned}
|\Psi_3^4\rangle \;=\;& |0000\rangle + |0112\rangle + |0221\rangle + \\
& |1011\rangle + |1120\rangle + |1202\rangle + \\
& |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

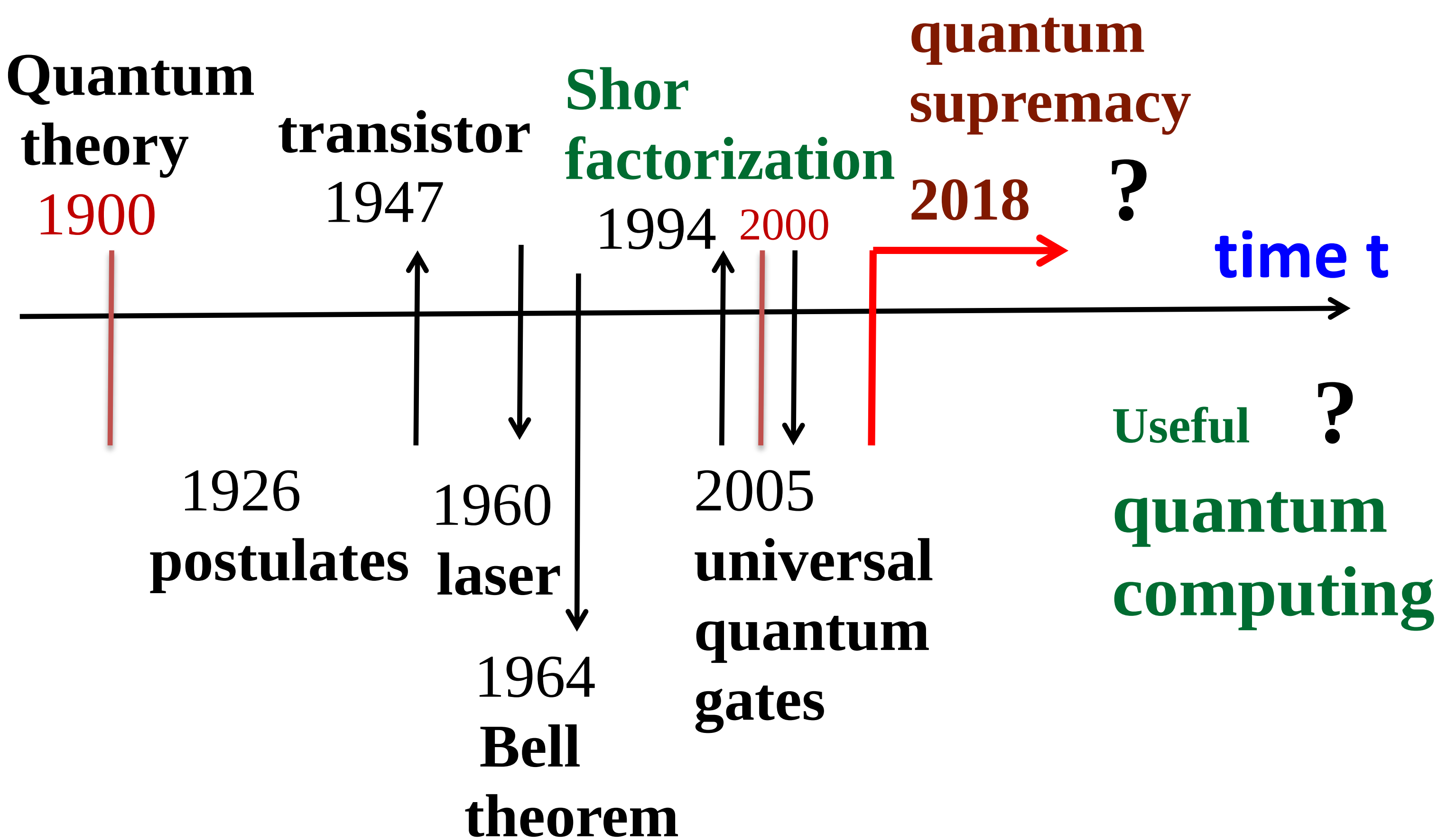
This state is also encoded in a pair of orthogonal Latin squares of size 3,

$$
\begin{array}{|c|c|c|}
\hline
0\alpha & 1\beta & 2\gamma \\
\hline
1\gamma & 2\alpha & 0\beta \\
\hline
2\beta & 0\gamma & 1\alpha \\
\hline
\end{array}
=
\begin{array}{|c|c|c|}
\hline
A\spadesuit & K\clubsuit & Q\diamondsuit \\
\hline
K\diamondsuit & Q\spadesuit & A\clubsuit \\
\hline
Q\clubsuit & A\diamondsuit & K\spadesuit \\
\hline
\end{array}
.
$$

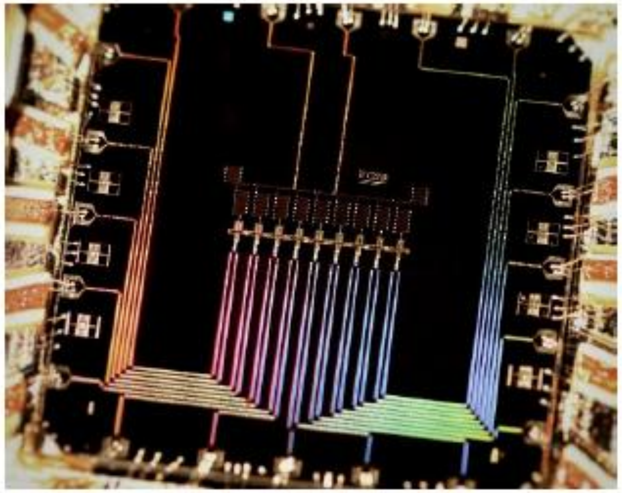
Corresponding **Quantum Code**:

$$
\begin{aligned}
|0\rangle &\to |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle \\
|1\rangle &\to |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle \\
|2\rangle &\to |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle
\end{aligned}
$$

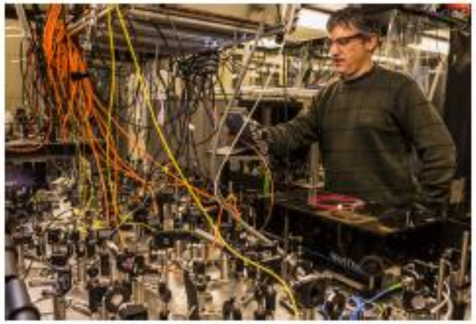
Time scale of the Second Quantum Revolution



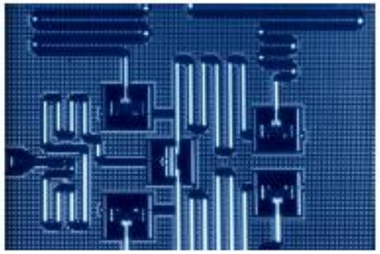
recent significant progress and fierce competition



Martinis (Google)



ionQ

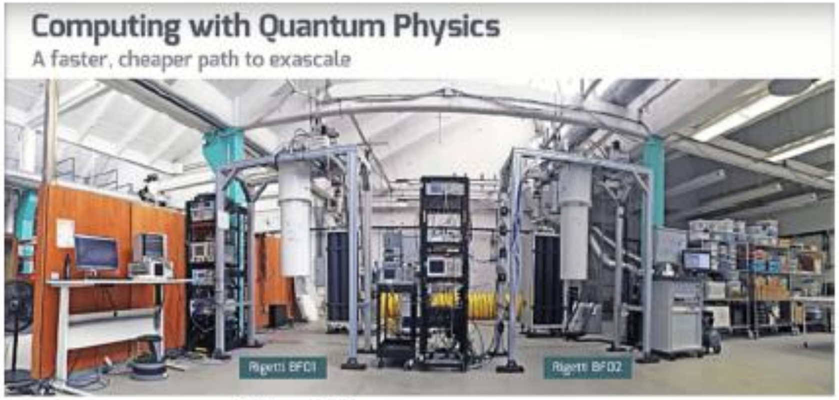


IBM cloud computer

STATION Q

Microsoft

DWAVE2



Rigetti

+LABS all over the world

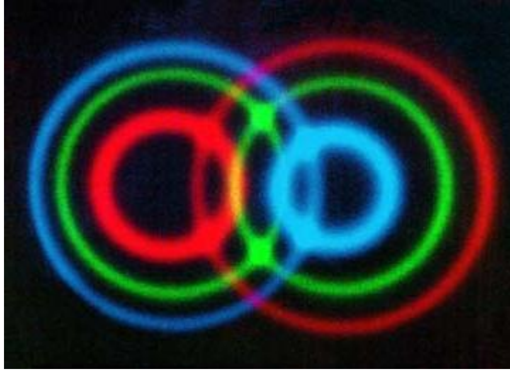
Wawel Castle in Cracow

Theorem of Danuta & Krzysztof Ciesielscy:

Ciesielscy theorem: with probability 1-ε the bench, Stefan Banach talked to Otto Nikodym in 1916 was located in η – neighborhood of the red arrow

Table memorizing the discussion of Banach and Nikodym

Concluding remarks



Quantum information evolves rapidly

- **new theoretical ideas are currently developed due to a constructive interference of physics, mathematics and computer science,**
- **fast progress of quantum technology allows us to witness their for practical implementation ,**
- **there are still several inspiring problems open :**

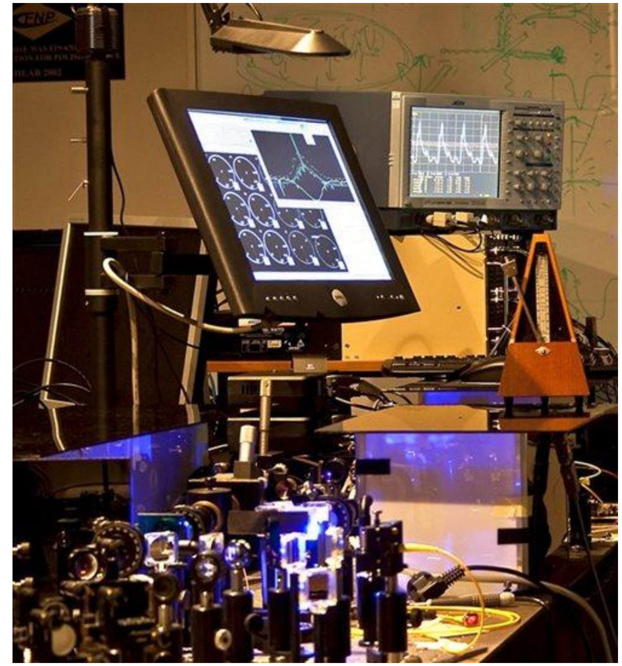
You are welcome to contribute to the field !

Quantum information in Poland

⟨KC|K⟩

National Center of Quantum Informatics

Sopot / Gdańsk University



National Lab of Atomic and Molecular Physics

Copernicus University, Toruń

theoretical groups in Cracow, Gliwice, Łódź, Katowice, Poznań, Warsaw, Wrocław…

Bench memorizing the discussion between

Otton Nikodym & Stefan Banacha (Cracow 1916)

sculpture: Stefan Dousa fot.: Andrzej Kobos

Cracow Planty Garden, close to the Wawel Castle

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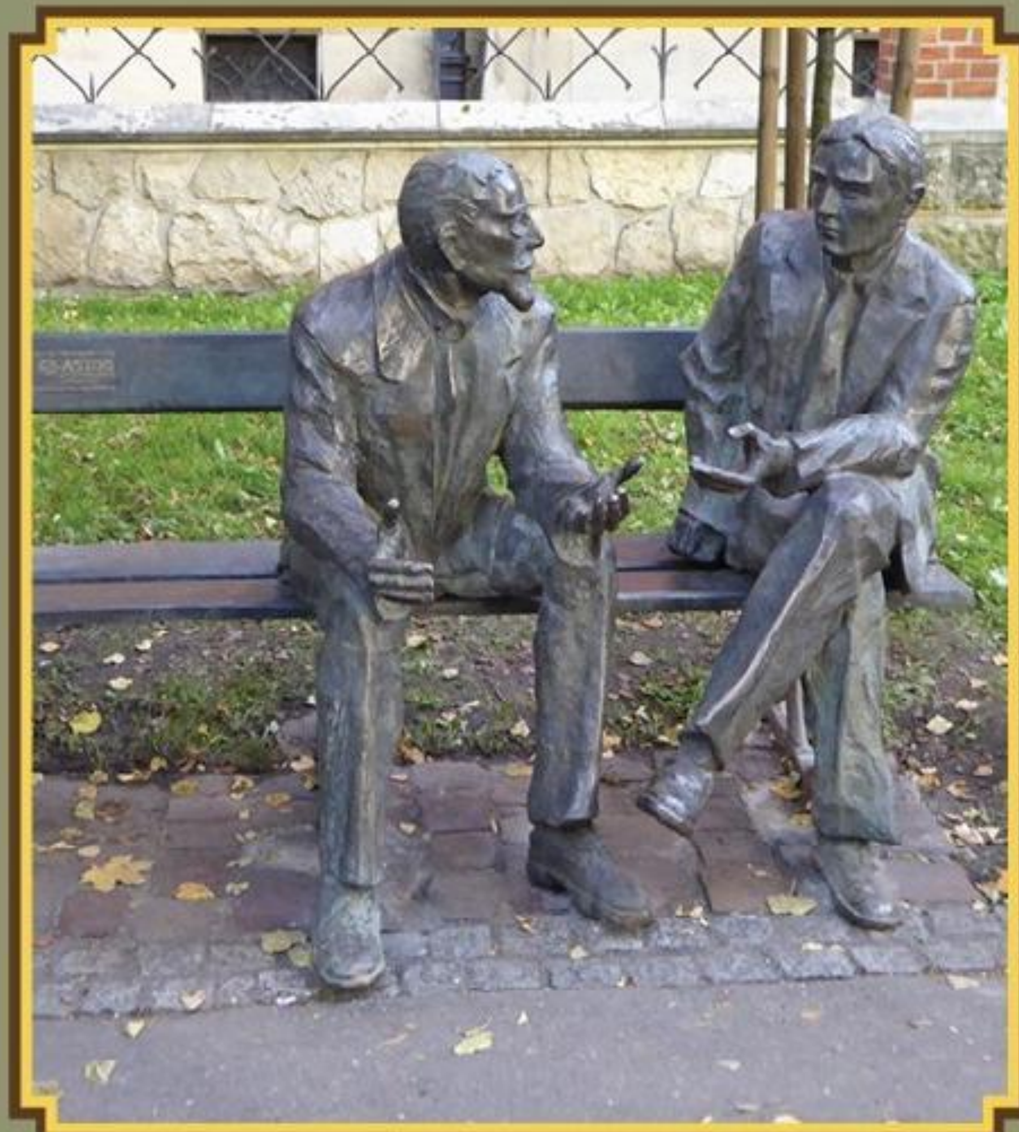
The Mathematical Intelligencer



Banach tells his side of the story