HANDLING IMPRECISION AND FLEXIBLE CONSTRAINTS IN VEHICLE ROUTING PROBLEMS: FUZZY APPROACH

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Abstract. The paper presents a mathematical model taking into account real constraints and goals of a concrete Polish transportation company. Although the authors started from a specific company situation their research led them to some general conclusions. The general model proposed takes into account such real features of transportation as flexibility of time constraints and uncertainty related to travel times. In the model both flexibility and uncertainty are handled by means of fuzzy sets. A procedure for scheduling vehicles under flexibility of constraints and uncertainty of travel times has been proposed and presented on an example.

1. Introduction

The importance of transportation in contemporary world cannot be overestimated. In every-day life as well as in economy, transportation is the basis of migration of people, goods and services. Confirmation of above truisms can be the number of vehicles in United States: more than 210 million, which is average 1.06 per person. The employment in transportation sector in US in 1993 was 8.5 million, coming in third, after medicine and education. ([4], [12]). According to [10], in Poland over 10% of sales revenues of products comes from sales of transport and storage.

Part of transportation business is the branch of actions of shipping companies, for which transport is the main source of income. The range of shippers' activities is wide: from shipping single cargo and receiving orders by phone, to planning shipping strategy and supply chain management.

Surprisingly, the market of computer aided shipping systems is not broad. When designers and architects have been using CAD systems for years and companies use integrated systems (e.g. SAP R/3, Max, Baan) for enterprise management, shippers still haven't adopted standards considering computer systems supporting them. Existing software is often written for specific company's situation and is useless in different context.

In [8] the author describes the practice of construction such systems: first, as routing algorithm some promising heuristic is chosen. During tests the algorithm works fine for most of the cases, but there are some instances for which the result is unacceptable. The heuristic is being improved leading to better, but more complicated algorithm. After a few such iterations the algorithm works good for required cases, but it is rather specialized and

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sensitive to changes in model – it will not work in different context: in other company or even in different geographic region.

When working on the system for one of the shipping companies (named TRANS from now on), we found ourselves in similar situation. Specificity of TRANS forced us to develop our own model of company. However, some of modeled concepts can be interesting to broader audience than only TRANS workers.

Because the main goal of a company is to make profit, its workers has to take care of cutting costs on orders being realized. On extreme, when order realization costs are higher than the order income, a shipper can deny realizing the order. However, when the customer making an order is considered an important one, shipper should accept such order despite the costs. Such specificity of TRANS was the main reason to develop custom enhancements to existing models.

There are many popular models of vehicle routing problem that could be a base for TRANS company. A new model is based on the model defined in [5].

The biggest flaw in existing systems is the lack of flexibility in modeling user's preference. The most popular aspect of preference being modeled are time windows. However, their representation is often simplest possible: time interval. When vehicle arrives in the interval, it's good. If not – it's bad. This binary interpretation was much to little in the case of TRANS company.

The assumption that everything in transportation goes according to a schedule is unrealistic. The route Warsaw-Berlin can take 8 hours, but can also take 12 if there is a queue on the Poland-Germany border. That conclusion caused us to think about considering uncertainty in TRANS routing model.

The use of fuzzy sets proved to be a powerful tool to modeling both uncertainty and preference. A variety of models using fuzzy sets theory in the OR literature is quite large. In the mids of models based on graph theory, linear programming and combinatorial optimization, a special attention should be paid to fuzzy scheduling problems [14]. The interesting fact observed in fuzzy scheduling models is that a fuzzy time parameter may represent an uncertainty or a preference. However depending on whether a given fuzzy time parameter represents an uncertainty or a preference the interpretation of membership function should be different and requires different treatment.

Dubois et al. [6] and Fortemps [9] shown that both uncertainty and preference modeled by means of fuzzy numbers in optimization problems can be handled together. Their works deal not only with methods and algorithms but with the interpretations of obtained results as well. This paper based on the previous achievements develops new concepts related to vehicle routing problems with flexible time windows and uncertain time parameters which is surprisingly poorly represented in the literature. Our considerations tend to a proposition of a scheduling algorithm that aims to propagate fuzzy driving times through fuzzy time windows. The procedure yields a schedule that is feasible for every realization of uncertain time parameters. Such an assumption requires a special care of succession of fuzzy time events. The solution procedure is based on the method previously implemented by Hapke and Słowiński [11] for fuzzy project scheduling problems.

The specificity of TRANS, described in Section 2 was the basis of such modifications. The generalization of a mathematical model as well as basic definitions are described in Section 3.

Sections 5 and 6 contain description of the model of preference and uncertainty in previously defined vehicle routing problem. A procedure for scheduling vehicles under flexibility of constraints and uncertainty of travel times is presented in Section 7.

Finally our conclusions and perspectives on future research are presented int Section 8.

2. The TRANS[1] company

The TRANS corporation is a shipping company from Poland. It owns the fleet of around 300 vehicles carrying cargo in Poland and Western Europe. The work of a shipper is not supported by any computer system.

Because of the high number of vehicles, one person is not able to schedule all the routes (a worker can handle scheduling of around 10 vehicles). The workers are split into groups of about 5 to 10 people. One group is assigned a fixed set of vehicles. Although the vehicles are not assigned to specific locations or routes between them, often the same vehicles are directed to the same region. They are because of the region-specific knowledge and experience of drivers.

Orders realized by the company are specified by a customer. Each order is characterized by the following attributes: the kind and amount of article to be transported, price (income) for realizing it, pickup and delivery places, and pickup and delivery *time windows*. The time windows describe when pickup or delivery actions can be taken. Arriving before time window start is allowed (the vehicle waits till the beginning of time window), but arriving after time window end is not allowed.

Customers are divided into strategic and casual groups. The strategic client is the one, whose orders have to be realized, regardless of costs. Orders from casual clients are realized only if they are profitable.

The *vehicles* being assigned to orders are split into three groups: own vehicles, contract vehicles and others' ones. Own vehicles are the property of the company and the company can assign them to any order it wants. Contract vehicles are the property of other shipping companies, with which TRANS has signed a contract for transporting a minimal amount of goods. Others' vehicles are used, when none of the rest of vehicles is capable of realizing the specific order (others' vehicles are not considered in the reminder of this paper).

The fleet of vehicles is heterogenous. Every vehicle is described by a set of attributes: a kind of article it can transport (e.g. palette, loose substance, liquids) and its maximal amount. During the decision process, the decision maker specifies (often unconsciously) the group of so called *candidate vehicles*, i.e. vehicles which can be considered as potential hauler for the specific order. This set is determined by the following factors:

- the vehicle can transport the kind of article specific to the order
- the vehicle is able to execute the order in required time windows
- the vehicle loading allows additional amount of goods to be carried
- the vehicle's driver has a permit to drive in countries specified in the order (this issue was not considered due to much complications it introduces to the model)

A vehicle can realize many orders in the same moment, of course only if it doesn't violate time and load capacity constraints. It allows the company for significant costs reduction.

The information about initial location of vehicles is known. It results from previous schedule and from info from the driver. Every driver is equipped with mobile phone and at

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1 In this paper the name of the company has been changed

any moment it is possible to check the vehicle location. It is planned to introduce GPS (global positioning system).

After the last planned order is finished, the vehicle does not go back to its base in Poland, but waits for another orders – unless it is desired to, for example for service. The awaiting place for future orders need not be the same as the delivery place for the last order. For example, let us assume that the last order in schedule ended in southern Italy. Experienced shipper knows, that in that region it is not uncommon to wait quite long for an order. Because of that it is often more appropriate to sent empty vehicle to the northern part of the country, where orders appear more frequently.

The criterion which guides the shipper in his scheduling efforts is rather simple: the best vehicle from candidate set is the one, which generates the smallest cost realizing the order. Because the income from the single order is constant, smaller costs mean higher profit. Additional aspect, which shipper may take into account is the satisfaction of the client.

2.1. Expectations from the system

The vehicle routing system is expected to support the shipper – it is not assumed that the system would help the shipper out in his daily work. It is mainly because the situation in shipping companies is very dynamic – sometimes occur unexpected situations, which were not anticipated during system modeling and require human to take up non standard actions. For example, the closure of the highway section requires update of input data, including travel times between places. It would be difficult to do it automatically, so human interaction is desirable.

Competence separation is determined by the kind of data existing in system. Static data, like vehicle parameters is in competence of the system. Dynamic data, like orders is in competence of the shipper, and he is responsible for providing it and keeping it up to date.

Moreover, it is not expected that the schedule found by the system will be perfect. It is the shipper who decides on the final schedule. The system is rather an adviser, than a dictator.

2.2. Input data

The system is fed with two kinds of data, static and dynamic. Static data are not changing during schedule search and can remain unchanged for several sessions. This data includes information about the fleet, places defined (i.e. the possible places of pickup and delivery) and distances between them, clients, regions and countries.

On the contrary, dynamic data concern the situation at the specific moment of the start of scheduling process. It includes information about orders, current schedules for vehicles, the next places to be visited by vehicles and respective dates.

3. Concepts and definitions

3.1. Basic VRP definition

There exist many different mathematical models of vehicle routing problem (VRP). The model which is the basis for our model has been proposed by Desrosiers et al. in [5]. Although there exist many simpler models (e.g. from [1] or [8]), the model from [5] is characterized by flexibility and easiness of modification.

Let $N=\{1, \ldots, n\}$ be the set of clients and $K=\{1, \ldots, k\}$ the set of vehicles available. For each $k \in K$ let us consider a graph $G^k=(V^k, A^k)$ consisting of the set of vertices (i.e. the places) and the set of arcs (i.e. the roads). V^k set contains elements of $N \cup \{o(k), d(k)\}$, where $o(k)$ denotes start point of the vehicle k (origin-depot) and $d(k)$ denotes end point of the vehicle k (destination-depot). A^k set contains allowed connections (roads) and is the subset of $V^k \times V^k$.

Each arc $(i,j) \in A^k$ is assigned a cost c_{ij}^k of its traversal. Each vehicle $k \in K$ is assigned its load capacity Q^k. Every client $i \in N$ requires pickup of ℓ_i units of goods and $\ell_{o(k)}$ describes initial load of vehicle k.

The problem defines two kinds of variables:

- flow variables X_{ij}^k, $(i,j) \in A^k$, $k \in K$ equal to 1, if the road (i,j) is traversed by the vehicle k and 0 otherwise
- load variables L_i^k, $i \in V^k$, $k \in K$ describing vehicle load after serving point i

Definition 1

The VRP is defined as follows:

$$\min \sum_{k \in K} \sum_{(i,j) \in A^k} c_{ij}^k X_{ij}^k \qquad (1)$$

s.t.

$$\sum_{k \in K} \sum_{j \in N \cup \{d(k)\}} X_{ij}^k = 1, \qquad \forall i \in N \qquad (2)$$

$$\sum_{j \in N \cup \{d(k)\}} X_{o(k),j}^k = 1, \qquad \forall k \in K \qquad (3)$$

$$\sum_{i \in N \cup \{o(k)\}} X_{ij}^k - \sum_{i \in N \cup \{d(k)\}} X_{ji}^k = 0, \qquad \forall k \in K, \forall j \in V^k \setminus \{o(k), d(k)\} \qquad (4)$$

$$\sum_{i \in N \cup \{o(k)\}} X_{i,d(k)}^k = 1, \qquad \forall k \in K \qquad (5)$$

$$X_{ij}^k (L_i^k + \ell_j - L_j^k) \leq 0, \qquad \forall k \in K, \forall (i,j) \in A^k \qquad (6)$$

$$\ell_i \leq L_i^k \leq Q^k, \qquad \forall k \in K, \forall i \in N \cup \{d(k)\} \qquad (7)$$

$$L_{o(k)}^k = \ell_{o(k)} \qquad \forall k \in K \qquad (8)$$

$$X_{ij}^{k} \in \{0, 1\} \qquad \forall k \in K, \forall (i,j) \in A^k \tag{9}$$

Equation (1) is the objective function, which is to minimize summary cost, including constant cost of vehicle usage.

Equation () ensures that every client is visited exactly once. Equations ()-(5) describe the flow on the path of vehicle k: vehicle leaves $o(k)$ exactly once, if arrives at any point, it also leaves that point and arrives $d(k)$ exactly once.

Constraints (6)-(8) describe vehicle's load size: at each place the vehicle has no less goods than the sum of the goods from the previous place and the amount of goods from the current place. In the depot $d(k)$ the vehicle has most goods, but no more than Q^K.

The condition (9) reflects binary nature of flow variables.

3.2. Pickup and delivery problem

Further literature search let us classify the model of TRANS company as "multiple vehicle pickup and delivery problem with time windows" (m-PDPTW) variant. In [5] authors claim this class of problems is rather poorly investigated. One significant solution algorithm is based on "branch and bound" method.

The differences between the previously defined VRP and m-PDPTW are as follows:

- The set N of all service places is divided into the set of pickup places $N^P=\{1, \ldots, n\}$ and the set of delivery places $N^D=\{1, \ldots, n\}$.
- The order i is defined as the transportation of p_i units of goods from place i to place $n+i$. Let $\ell_i=p_i$ for $i \in N^P$ and $\ell_{n+i}=-p_i$ for $n+i \in N^D$. The ℓ_i value is interpreted as the change (positive or negative) of vehicle load after visiting place i.
- Besides the cost of traversal c_{ij}^{k}, $(i,j) \in A^k$, $k \in K$, there exists time of traversal from i to j, t_{ij}^{k}, including serving time at point i.
- For every $i \in N$ there exists time window $[a_i, b_i]$, within which the service must start. Because the vehicle k might realize previous orders, we assume the initial time window at $o(k)$, $[a_{o(k)}, b_{o(k)}]$ and initial load $\ell_{o(k)}$. We also define final time window at $d(k)$, $[a_{d(k)}, b_{d(k)}]$.
- In addition to flow variables and load variables there exist time variables T_i^k, $i \in V^k$, $k \in K$, which describe the time of service start in place i by vehicle k. If the vehicle does not visit place i, the value of T_i^k is irrelevant.

So extended problem can be written as:

Definition 2 (Problem 1)

$$\min \sum_{k \in K} \sum_{(i,j) \in A^k} c_{ij}^{k} X_{ij}^{k} \tag{10}$$

s.t.

$$\sum_{k\in K}\sum_{j\in N\cup\{d(k)\}} X_{ij}^{k}=1, \qquad \forall i\in N^{P} \qquad (11)$$

$$\sum_{j\in N^{P}\cup\{d(k)\}} X_{o(k),j}^{k}=1, \qquad \forall k\in K \qquad (12)$$

$$\sum_{i\in N\cup\{o(k)\}} X_{ij}^{k} - \sum_{i\in N\cup\{d(k)\}} X_{ji}^{k}=0, \qquad \forall k\in K, \forall j\in N \qquad (13)$$

$$\sum_{i\in N^{D}\cup\{o(k)\}} X_{i,d(k)}^{k}=1, \qquad \forall k\in K \qquad (14)$$

$$X_{ij}^{k}(T_{i}^{k}+t_{ij}^{k}-T_{j}^{k})\leq 0, \qquad \forall k\in K, \forall (i,j)\in A^{k} \qquad (15)$$

$$a_{i}\leq T_{i}^{k}\leq b_{i}, \qquad \forall k\in K, \forall i\in V^{k} \qquad (16)$$

$$X_{ij}^{k}(L_{i}^{k}+\ell_{j}-L_{j}^{k})=0, \qquad \forall k\in K, \forall (i,j)\in A^{k} \qquad (17)$$

$$\ell_{i}\leq L_{i}^{k}\leq Q^{k}, \qquad \forall k\in K, \forall i\in N^{P} \qquad (18)$$

$$0\leq L_{n+i}^{k}\leq Q^{k}-\ell_{i}, \qquad \forall k\in K, \forall n+i\in N^{D} \qquad (19)$$

$$L_{o(k)}^{k}=\ell_{o(k)} \qquad \forall k\in K \qquad (20)$$

$$T_{i}^{k}+t_{i,n+i}^{k}-T_{n+i}^{k}\leq 0 \qquad \forall k\in K, \forall i\in N^{P} \qquad (21)$$

$$\sum_{j\in N} X_{ij}^{k}-\sum_{j\in N} X_{j,n+i}^{k}=0, \qquad \forall k\in K, \forall i\in N^{P} \qquad (22)$$

$$X_{ij}^{k}\in\{0,1\} \qquad \forall k\in K, \forall (i,j)\in A^{k} \qquad (23)$$

Equation (10) is the objective function, which is to minimize summary cost. Further in the paper we will modify this function to incorporate other aspects, like orders' income or awaiting costs.

Equation (11) guarantees that every place of pickup i is visited exactly once, just by one vehicle. In conjunction with (22) (which describes that if vehicle visits some pickup place i, it also has to visit corresponding delivery place $n+i$) it assures us that every order is realized by exactly one vehicle.

The formulas (12)-(14) define so-called flow conditions on the values of X_{ij}^{k}. In the context of vehicle routing it can be seen as the rules for vehicle routes: each vehicle k leaves $o(k)$, visits some number of places (each no more than once) and ends in place $d(k)$.

Conditions (15)-(16) restrict the time in which vehicles has to serve each place they visit. In each place the service has to start within the appropriate time window. Moreover, next client can be served only after serving its direct predecessor is finished.

The formulas (17)-(20) form conditions on the load of each vehicle. At one moment vehicle's maximum capacity cannot be exceeded and the total load of goods must not be lower than the sum of weights of picked and not yet delivered goods.

Inequality (21) assures proper order of pickup and delivery in time. In every realized order, the start time of pickup must be early enough before the start of delivery to let the vehicle move from pickup place to delivery place.

Again, the formula (23) describes binary nature of flow variables.

3.3. Flexible constraints

Time windows concept used in VRP definitions is most commonly understood as time interval. The vehicle is required to arrive before the end of the interval. Arriving before the start of the interval causes additional waiting for service start, till the beginning of the interval.

This model, however, does not correspond to real situations. According to the model, arriving even 5 minutes after the end of interval makes the solution infeasible. In practice, the customer can accept small lateness, even if it makes him less satisfied.

Above observation is not a novelty – many authors have built models incorporating customer's partial satisfaction. Most popular, especially on the area of scheduling, are Zimmermann ([15]), Dubois and Prade ([6]) and Fortemps ([9]). The model proposed in this paper is based on works of these authors.

In a case of crisp time windows a satisfaction level can be measured by a binary function that takes values from $\{0, 1\}$. Value 1 means full satisfaction (e.g. arrival was on time), where 0 means dissatisfaction (e.g. arrival occurred after the end of time window). If one wanted to express a partial satisfaction it could be measured by a value within [0, 1].

For instance assume that in fact customer is fully satisfied if arrival is before 15:00, but can also accept lateness of no more than 15 minutes. Then, the satisfaction level of customer is equal to 1 for times earlier than 15:00 and equal to 0 for times later 15:15. And what about the times between 15:00 and 15:15? We can extend the satisfaction level definition by allowing it to take any values from the interval [0,1]. Like in a case of crisp time windows, value 0 means dissatisfied customer, value 1 means fully satisfied client. Moreover, the closer the satisfaction level to 1, the more satisfied client is.

Later in the paper time windows which are not simple intervals, but are described by satisfaction level functions, are called *flexible time windows*.

3.4. Fuzzy sets

The definition of satisfaction level function as the convex function with values within [0,1] can also be seen as a membership function of a fuzzy set ([6]). In the example described above, the fuzzy set models the values of arrival times accepted by the customer.

Definition 3

The *fuzzy set* $\tilde{A}$ on a universe X is defined as the set of ordered pairs:

$$\tilde{A}=\{(x,\mu_{\tilde{A}}(x))|x\in X\} \tag{24}$$

where $\mu_{\tilde{A}}:X\rightarrow[0,1]$ is the *membership function* of the fuzzy set $\tilde{A}$.

In the paper, the notation used to describe fuzzy constraint $\widetilde{A}$ (e.g. flexible time window) is as follows:

$$\widetilde{A}=(\underline{a},\underline{b},\overline{b},\overline{a}) \tag{25}$$

where $(\underline{a},\overline{a})$ is the interval of non-zero satisfaction level and $(\underline{b},\overline{b})$ is the interval of satisfaction level equal to 1 (figure 1)

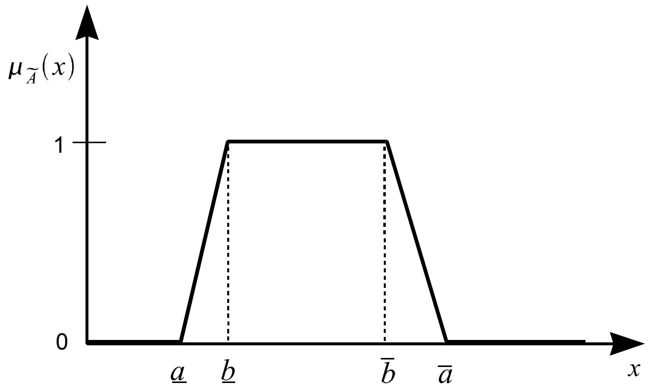

Figure 1. 4-points representation of fuzzy interval

3.5. Defuzzification and comparison of fuzzy numbers

In optimization the results obtained have to be evaluated and compared in order to select best alternatives. Comparing results of fuzzy optimization is far more difficult than comparing two real numbers and can bring ambiguity.

In the literature there are many methods for comparison of fuzzy numbers – the survey can be found in [3]. The most common way of comparing fuzzy values $\widetilde{A}$ and $\widetilde{B}$ is to define a real function *F* (defuzzification function) on the fuzzy set, and to compare two real values, $F(\widetilde{A})$ and $F(\widetilde{B})$. Needless to say, how important the selection of *F* is.

In the simplest case, the function *F* is selected based on some characteristics of fuzzy numbers, like the element with highest membership value, weighted average or center of gravity. Proposed in [13] and extended in [9] the method of area compensation has some advantages over other methods and was used in TRANS vehicle routing system.

Let us define two values (figure 2):

$$S_L(\widetilde{A}\geq\widetilde{B})=\int_{U(\widetilde{A},\widetilde{B})}[\underline{a}_\alpha-\underline{b}_\alpha]\,\mathrm{d}\,\alpha \tag{26}$$

$$S_R(\widetilde{A}\geq\widetilde{B})=\int_{V(\widetilde{A},\widetilde{B})}[\overline{a}_\alpha-\overline{b}_\alpha]\,\mathrm{d}\,\alpha \tag{27}$$

where

$$U(\widetilde{A},\widetilde{B})=\{\alpha\,|\,\alpha\in[0,1]\wedge\underline{a}_\alpha\geq\underline{b}_\alpha\} \tag{28}$$

$$V(\widetilde{A},\widetilde{B})=\{\alpha\,|\,\alpha\in[0,1]\wedge\overline{a}_\alpha\geq\overline{b}_\alpha\} \quad (29)$$

$$\underline{a}_\alpha=\inf_{x\in\mathbb{R}}\{x:\mu_{\widetilde{A}}(x)\geq\alpha\} \quad (30)$$

$$\overline{a}_\alpha=\sup_{x\in\mathbb{R}}\{x:\mu_{\widetilde{A}}(x)\geq\alpha\} \quad (31)$$

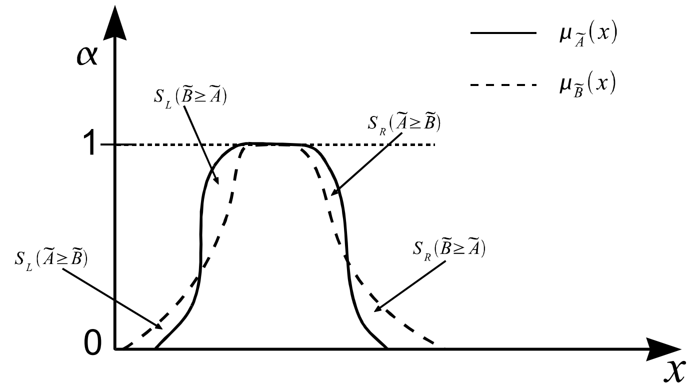

Figure 2. Fuzzy numbers comparison – area compensation method

Interpretation of $S_L(\widetilde{A}\geq\widetilde{B})$ is as follows: it is equal to the area bounded by left slopes of fuzzy numbers, where the slope of $\widetilde{A}$ is on the right of the slope of $\widetilde{B}$ (i.e. where $\underline{a}_\alpha\geq\underline{b}_\alpha$. By analogy, $S_R(\widetilde{A}\geq\widetilde{B})$ is interpreted as the area bounded by right slopes, where the slope of $\widetilde{A}$ is on the right of the slope of $\widetilde{B}$. The value of

$$C(\widetilde{A}\geq\widetilde{B})=\frac{1}{2}(S_L(\widetilde{A}\geq\widetilde{B})+S_R(\widetilde{A}\geq\widetilde{B})-S_L(\widetilde{B}\geq\widetilde{A})-S_R(\widetilde{B}\geq\widetilde{A})) \quad (32)$$

measures the advantage of "areas for $\widetilde{A}$ " over "areas for $\widetilde{B}$ ". Furthermore, we can assume:

$$\widetilde{A}\geq\widetilde{B},\ \text{if}\ C(\widetilde{A}\geq\widetilde{B})\geq 0 \quad (33)$$

$$\widetilde{A}>\widetilde{B},\ \text{if}\ C(\widetilde{A}\geq\widetilde{B})>0 \quad (34)$$

$$\widetilde{A}\sim\widetilde{B},\ \text{if}\ C(\widetilde{A}\geq\widetilde{B})=0 \quad (35)$$

As shown by Fortemps in [9]:

$$C(\widetilde{A}\geq\widetilde{B})=\Im(\widetilde{A})-\Im(\widetilde{B}) \quad (36)$$

where

$$\Im(\tilde{A}) = \frac{1}{2}\int_0^1 (\underline{a}_\alpha + \overline{a}_\alpha)\,\mathrm{d}\,\alpha \tag{37}$$

is interpreted as the mean value of fuzzy number, with respect to the Dempster-Shafer theory. From the equations (33)-(36) we can conclude that the function $\Im(\tilde{A})$ can be understood as the defuzzified value of $\tilde{A}$.

As the objective function (74) being optimized in this paper is fuzzy we apply the method described above for comparison of fuzzy results.

3.6. Fuzzy linear programming

The problem definition introduced in section 3.2 and developed through the rest of the paper uses standard form of mathematical programming model. Zimmermann in [15] defined fuzzy modification of the linear programming model, where the objective function

$$\min\ c_1 x_1 + \ldots + c_n x_n \tag{38}$$

is converted into additional constraint:

$$c_1 x_1 + \ldots + c_n x_n \leq b_0 \tag{39}$$

where b_0 is predefined *aspiration level*. Further, Zimmermann introduces fuzzy version of inequality operator, as the satisfaction level of the constraint by the value of *x*. If *i*-th constraint has the form:

$$a_{i1} x_1 + \ldots + a_{¿} x_n \leq b_i \tag{40}$$

than its fuzzy representation with membership value $\mu_i(x)$ has the form as shown on figure 3

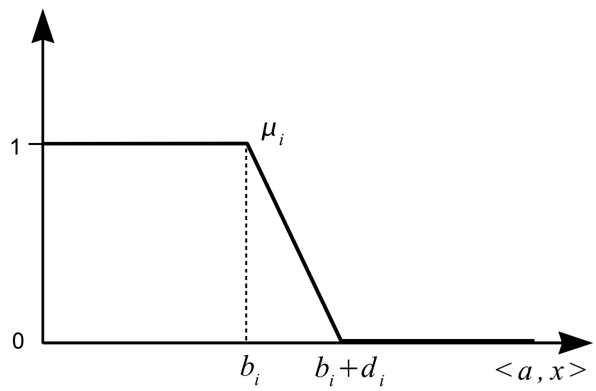

Figure 3. Fuzzy form of constraint in fuzzy linear programming

The fuzzy decision is then defined by Bellman and Zadeh's principle [2] as

$$\mathrm{Sat}(x) = \min\{\mu_0(x), \ldots, \mu_m(x)\} \tag{41}$$

and the optimal solution to the fuzzy linear programming problem is determined from:

$$\mathrm{Sat}(x^*) = \max_x \ \mathrm{Sat}(x) \tag{42}$$

4. Problem definition

In section 3.1 the basic model of VRP has been described. As mentioned before it has to be adopted to incorporate specific needs of TRANS company. For this purpose we extend m-PDPTW model from definition 2 with not realized orders and orders' income.

Because every company's main goal is to maximize profits, the criterion of cost has been exchanged with the total profit coming from realization of orders. However, because the total income is constant, the problem of finding the greatest profit is equivalent to the problem of finding the smallest cost.

The other incompatibility with company's reality is the necessity of realization of all the orders. It is not hard to find some difficult orders, realization of which costs too much. Better than that, there occur infeasible orders, because of too small gap between required pickup and delivery time windows. Moreover, in reality the shipper can deny realizing the order if it is too expensive for him.

Realization of only part of orders implies that equivalence of minimizing the cost and maximizing the profit is no longer valid. Thus we need to modify the model. Notice that the number of constraints in some conditions (11)-(23) depends on the number of orders. It would be undesirable to define the number of constraints as the function of the number of **realized** orders. To avoid that, let us assume the realization of every order, but the orders which we do not want to realize are assigned to a dummy vehicle k=0. The route of this vehicle should have special properties:

- any order can be added to or removed from the route without the loss of solution correctness
- no order on the route has any direct impact on objective function

The above can be implemented in two ways:

- by appropriate construction of the route for vehicle k=0 (e.g. zero costs and times of travel between any two points) and additional assumptions about the vehicle (e.g. unbounded load capacity)
- by skipping vehicle k=0 in all possible constraints

Although the first solution is more systematic, we opted for the second one – it is simpler and imposes less additional constraints.

The constraints which must be defined for vehicle k=0 are only (11) (every order is realized by exactly one vehicle), (22) (the same vehicle which picked up a cargo has to deliver it) and (23) (flow variables are binary). For vehicle k=0 the route is then no longer “real” route, as for other vehicles (only the arcs between places from every assigned order have to be traversed). It merely tells us, that the orders *i* for which:

$$\sum_{j\in N} X_{ij}^{0}=1 \tag{43}$$

are considered not realized.

Let *K* will denote the set of existing vehicles, as previously. Let K^0 be the set of all vehicles including the dummy, $K^0=K\cup\{0\}$. Then appropriate modification are as follows:

$$\sum_{k\in K^0}\ \sum_{j\in N\cup\{d(k)\}} X_{ij}^{k}=1, \qquad \forall i\in N^P \tag{44}$$

$$\sum_{j\in N} X_{ij}^{k}-\sum_{j\in N} X_{j,n+i}^{k}=0, \qquad \forall k\in K^0, \forall i\in N^P \tag{45}$$

$$X_{ij}^{k}\in\{0,1\} \qquad \forall k\in K^0, \forall (i,j)\in A^k \tag{46}$$

As the objective function (10) summarizes costs for vehicles in *K*, the vehicle *k*=0 has no impact on its value. Also, no order assigned to vehicle *k*=0 has any impact on any other vehicle's route.

As mentioned earlier, customers are split into two groups: strategic ones and casual ones. The strategic ones are characteristic, because their orders have to be realized. We can enforce such behavior in the model by introducing additional set of parameters:

$$St_i=\begin{cases}1, \text{ if the order } i \text{ comes from strategic client}\\ 0, \text{ otherwise}\end{cases} \tag{47}$$

and adding new constraint

$$St_i \sum_{j\in N\cup\{d(0)\}} X_{ij}^{0}=0 \qquad \forall i\in N^P \tag{48}$$

which says that no order from strategic client can be assigned to vehicle *k*=0.

To take into account the income from realized orders we modify the objective function (10). Let inc_i be the income from realization of order *i*. Then the modified objective function is as follows:

$$\max \sum_{k\in K}\left(\sum_{i\in N^P}\left(inc_i \sum_{j\in V^k} X_{ji}^{k}\right) - \sum_{(i,j)\in A^k} c_{ij}^{k} X_{ij}^{k}\right) \tag{49}$$

Here, the cost criterion has been changed to the gain criterion. The gain (i.e. profit) is defined as the difference between the income from the order and the cost of realizing the order. The formula (49) sums the total profit of all vehicles up (except for *k*=0). A profit for each vehicle is calculated as the sum of all realized orders minus the sum of all costs of the route.

Definition 4

The *route* of vehicle k described as $R^k=(r_1^k, r_2^k, \dots, r_{rl^k}^k)$ is the longest path in graph $G^k=(V^k, A^k)$, such that

$$X_{r_j^k, r_{j+1}^k}=1 \qquad \forall\, j\in\{1, \dots, rl^k-1\} \tag{50}$$

It is also imposed by constraints of definition 2, that

$$r_1^k=o(k) \tag{51}$$

$$r_{rl^k}^k=d(k) \tag{52}$$

Simply speaking, the route is the sequence of places visited by the vehicle, ordered by the time of visit.

5. Introducing flexible constraints

As mentioned previously, shipper's intuition is the most difficult aspect to model. As of today we have no better ways of incorporating it than description with strict and limited rules.

One of the factors that influences shipper's intuitive behavior is the conviction that he has to maximize clients' satisfaction – satisfied client means more orders in the future and more income. Client satisfaction can be reached by many means: by visiting places at scheduled dates, ensuring high-quality service and fast deliveries. In this article the model of the most important aspect of clients' satisfaction is described: arrivals on time, within time windows. Although time windows appear in VR problems, their usual representation as time interval does not correspond to real situation – service start even 5 minutes after the interval end is unacceptable, despite of usual customer's agreement on such situation. Below such compromise is proposed, modeled using fuzzy intervals.

5.1. Satisfaction level

Let $\mu_{\tilde{A}}(x)$ be the satisfaction level of the client, if the service starts at time x. When $\mu_{\tilde{A}}(x)=1$ we can say that the client is fully satisfied, when $\mu_{\tilde{A}}(x)=0$, it can be interpreted as the full dissatisfaction of the customer. In the classic implementation of time windows, the set of values for which $\mu_{\tilde{A}}(x)=1$ is represented as the interval (figure 4).

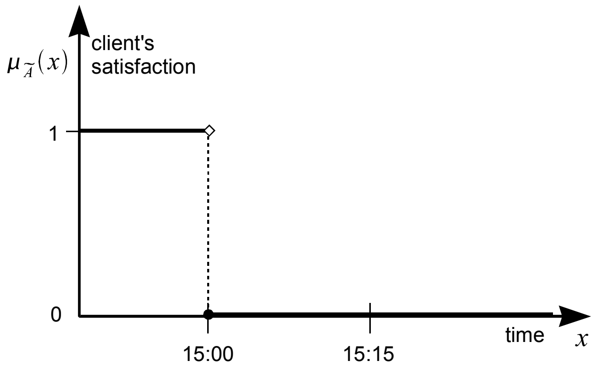

Figure 4. Client's satisfaction level– unreal precision

However, only 2 minutes distance, from 14:59 to 15:01 marks drastic drop of client's satisfaction, from 1 to 0. Of course, this is unreal situation. Better than that would be the model, where small overdue causes small loss of customer's satisfaction. This is presented on figure 5.

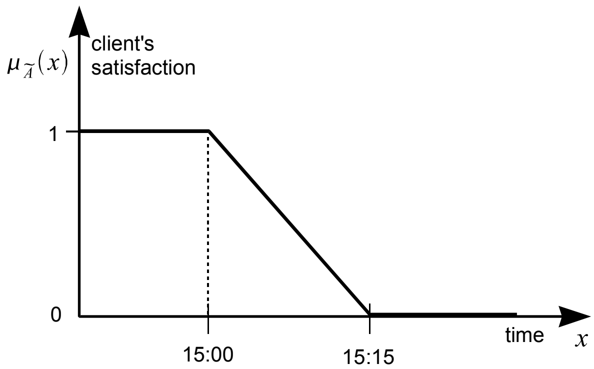

Figure 5. Client's satisfaction level – realistic situation

Here, not only values 0 and 1 are present in the set of possible client's satisfaction values. All numbers in between represent "partial" satisfaction: the closer $\mu_{\tilde{A}}(x)$ is to 1, the more satisfied is the client. As described in section 3.3 the value of $\mu_{\tilde{A}}(x)$ can be interpreted as the membership function value for the fuzzy set of service times which satisfy client (customer satisfaction degree).

The client's satisfaction is of the "not later than" type, representing the right bound of time window. On the other hand, the left bound is of different nature. "Not earlier than" constraint results from the following conditions:

- service points (e.g. shops, warehouses, wholesalers) start working at the strictly specified moment
- the shipping company does not want its vehicles to wait too long

Because of that, left bound on the service start time is considered rigid and represents shipper's satisfaction. For the sake of model's uniformity this can also be modeled as the fuzzy set, whose slope is very steep (figure 6) – vertical in the limit.

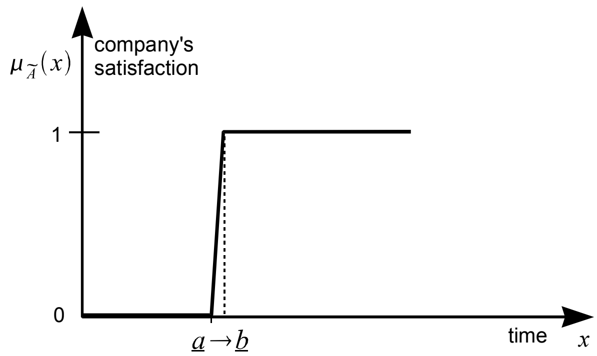

Figure 6. Company's satisfaction – very steep slope

To simplify calculations we can join two types of constraints described above – of course only for the same place. Joined constraints determine time window as fuzzy interval (figure 7). We call such a constraint a *flexible constraint* (see section 3.3).

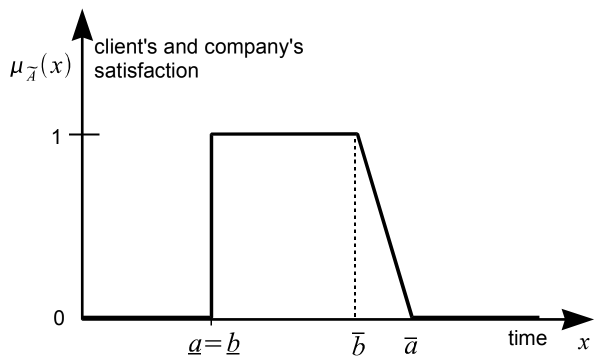

Figure 7. Joined satisfaction

Let $\tilde{C}_i=(\underline{a}_i,\underline{b}_i,\overline{b}_i,\overline{a}_i)$ be the flexible constraint of service start at place *i*. Additionally, $\tilde{C}_i^R=(-\infty,-\infty,\overline{b}_i,\overline{a}_i)$ and $\tilde{C}_i^L=(\underline{a}_i,\underline{b}_i,\infty,\infty)$ define client's and company's satisfaction, respectively. If vehicle *k* arrives at this place in the moment T_i^k, the satisfaction level is equal to membership value $\mu_{\tilde{C}_i}(T_i^k)$. If we would like to keep definition 2 coherent, we could replace constraint (16) with the following one:

$$\mu_{\tilde{C}_i}(T_i^k)\geq\alpha_i \qquad \forall k\in K, \forall i\in V^k \tag{53}$$

for chosen set of α_i -cuts. But using such a transformation of flexible time windows to crisp ones, all the information about actual satisfaction level at every single place would be lost – we would only know, whether all constraints were met (i.e. the solution to the whole problem has been found) or not.

Much better solution is to remove constraint (16) from definition 2 and add additional criterion expressing the global satisfaction level.

Definition 5

Let $C=\{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_{2n}\}$ be the set of flexible constraints on service starts in 2*n* places. For every solution *x* of problem 1 without constraint (16), the *global satisfaction level* of *C*, $\text{Sat}(x)$ is defined as:

$$\text{Sat}(x)=\mu_{\widetilde{C}_1 \cap \ldots \cap \widetilde{C}_{2n}}(x)=\min_{i \in N, k \in K} \mu_{\widetilde{C}_i}(T_i^k) \tag{54}$$

In other words, the global satisfaction level is the satisfaction level of the least satisfied flexible constraint (see section 3.3).

Definition 6

The problem 1, without the constraint (16) and with another criterion:

$$\max \ \text{Sat}(x) \tag{55}$$

is *multiobjective fuzzy vehicle routing problem*.

The criterion (55) does not reject a solution, with $\text{Sat}(x)=0$ (in general it could be Pareto-optimal). To guarantee the fulfillment of all flexible constraints, inequalities (53) can be added, with non-zero cut-levels α_i.

Another method is fuzzy linear programming, proposed by Zimmermann and shortly described in section 3.6. Using this approach, objective (49) is converted into additional constraint:

$$\sum_{k \in K} \left(\sum_{i \in N^P} (inc_i \sum_{j \in V^k} X_{ji}^k) - \sum_{(i,j) \in A^k} c_{ij}^k X_{ij}^k \right) \geq b_0 \tag{56}$$

where b_0 is aspiration level. It can be troublesome to find aspiration level close enough to optimal value. Shipper's assistance or metaheuristic search can be helpful.

5.2. CPM with flexible constraints

In scheduling, where fuzzy sets have been widely adopted, the basic operation used to verify schedule feasibility is *critical path method* (CPM). It is based on propagation of time constraints along the "execution path" of the schedule.

The *earliest starting time* of a task, *est*, is determined by the *earliest finish times* of all preceding tasks, *eft*: the task cannot start before the last of preceding ones ends. The *latest finish time* of a task, *lft* is determined by the *latest starting times* of all following tasks, *lst*: the task must end, so all the following tasks manage to finish on time. These assumptions can be written as:

$$est_i \leq s_i \leq lft_i - t_i \qquad \forall i \tag{57}$$

$$est_i \geq \max_{j: O_j \to O_i} (est_j + t_j) \qquad \forall i \tag{58}$$

$$lft_i \leq \min_{j:O_i \to O_j} (lft_j - t_j) \qquad \forall i \qquad (59)$$

where t_i is the execution time of task O_i, s_i its start and → the precedence relation.

In the first phase of CPM, the constraint (58) propagates, starting with tasks with no predecessors. If for every task the earliest starting time has been fixed, phase two of CPM begins. The constraint (59) propagates backward, starting with tasks with no successors. The propagation steps use the following formulas:

$$est_i = \max\{est_i, \max_{j:O_j \to O_i} (est_j + t_j)\} \qquad \forall i \qquad (60)$$

$$lft_i = \min\{lft_i, \min_{j:O_i \to O_j} (lft_j - t_j)\} \qquad \forall i \qquad (61)$$

Similarly, in VRP, earliest starting time and latest finishing time can be concluded from the values $\widetilde{C}^L$ and $\widetilde{C}^R$, respectively. For every α-cut, we can define following constraints corresponding to constraints (57)-(59):

$$(\widetilde{C}_i^L)_\alpha \leq T_i^k \leq (\widetilde{C}_i^R)_\alpha \qquad \forall k, \forall i \in V^k : \sum_{j \in V^k} X_{ji} = 1 \qquad (62)$$

$$(\widetilde{C}_j^L)_\alpha \geq (\widetilde{C}_i^k)_\alpha^L + t_{ij}^k \qquad \forall k, \forall (i,j) \in A^k, X_{ij} = 1 \qquad (63)$$

$$(\widetilde{C}_i^R)_\alpha \leq (\widetilde{C}_j^R)_\alpha - t_{ij}^k \qquad \forall k, \forall (i,j) \in A^k, X_{ij} = 1 \qquad (64)$$

Of course not every value of α meets the constraints (62)-(64) – the greatest of values of α is equal to the global satisfaction level, $\mathrm{Sat}(x)$.

Equations (60)-(61) have equivalent ones in the terms of time windows:

$$(\widetilde{C}_j^L)_\alpha = \max\{(\widetilde{C}_j^L)_\alpha, (\widetilde{C}_i^L)_\alpha + t_{ij}^k\} \qquad \forall k, \forall (i,j) \in A^k, X_{ij} = 1 \qquad (65)$$

$$(\widetilde{C}_i^R)_\alpha = \min\{(\widetilde{C}_i^R)_\alpha, (\widetilde{C}_j^R)_\alpha - t_{ij}^k\} \qquad \forall k, \forall (i,j) \in A^k, X_{ij} = 1 \qquad (66)$$

Using extension principle, equations (65)-(66) can be extended to the flexible constraints:

$$\widetilde{C}_j^L = \max\{\widetilde{C}_j^L, \widetilde{C}_i^L \oplus t_{ij}^k\} \qquad \forall k, \forall (i,j) \in A^k, X_{ij} = 1 \qquad (67)$$

$$\widetilde{C}_i^R = \min\{\widetilde{C}_i^R, \widetilde{C}_j^R \ominus t_{ij}^k\} \qquad \forall k, \forall (i,j) \in A^k, X_{ij} = 1 \qquad (68)$$

Calculation of feasible, fuzzy time windows is the same as for classical CPM: first, equation (67) is propagated along route of vehicle k, then equation (68) is propagated backward.

Example 1

Assume, that there is a route with three places. Initial values of $\tilde{C}_i^R$, $\tilde{C}_i^L$ and t_{ij} are given in the table:

i	1	2	3
$\tilde{C}_i^L$	(1, 2, ∞, ∞)	(7, 8, ∞, ∞)	(13, 13.5, ∞, ∞)
$\tilde{C}_i^R$	(-∞, -∞, 3.5, 4)	(-∞, -∞, 10, 10.8)	(-∞, -∞, 15.5, 16)
$t_{i,i+1}$	4.5	8	-

Let us trace the propagation steps. First, the propagation of (67):

$\tilde{C}_1^L=(1, 2, \infty, \infty)$

$\tilde{C}_2^L=\max\{\tilde{C}_2^L, \tilde{C}_1^L+t_{12}\}=\max\{(7, 8, \infty, \infty), (5.5, 6.5, \infty, \infty)\}=(7, 8, \infty, \infty)$

$\tilde{C}_3^L=\max\{\tilde{C}_3^L, \tilde{C}_2^L+t_{23}\}=\max\{(13, 13.5, \infty, \infty), (15, 16, \infty, \infty)\}=(15, 16, \infty, \infty)$

Then, the propagation of (68):

$\tilde{C}_3^R=(-\infty, -\infty, 15.5, 16)$

$\tilde{C}_2^R=\min\{\tilde{C}_2^R, \tilde{C}_3^R-t_{23}\}=\min\{(-\infty, -\infty, 10, 10.8), (-\infty, -\infty, 7.5, 8)\}=(-\infty, -\infty, 7.5, 8)$

$\tilde{C}_1^R=\min\{\tilde{C}_1^R, \tilde{C}_2^R-t_{12}\}=\min\{(-\infty, -\infty, 3.5, 4), (-\infty, -\infty, 3, 3.5)\}=(-\infty, -\infty, 3, 3.5)$

The figure 8 shows graphically time windows received in the above examples. It can be seen that not all constraints can be met with satisfaction equal to 1: for such cut level, the earliest starting time is later than latest starting time in constraints $\tilde{C}_2$ and $\tilde{C}_3$. Only for $\alpha_2=\alpha_3=0.67$ there exists feasible solution, e.g. $(T_1, T_2, T_3)=([1..2], 7.67, 15.67)$ with satisfaction levels $(\mu_{\tilde{C}_1}(T_1), \mu_{\tilde{C}_2}(T_2), \mu_{\tilde{C}_3}(T_3))=(1, 0.67, 0.67)$.

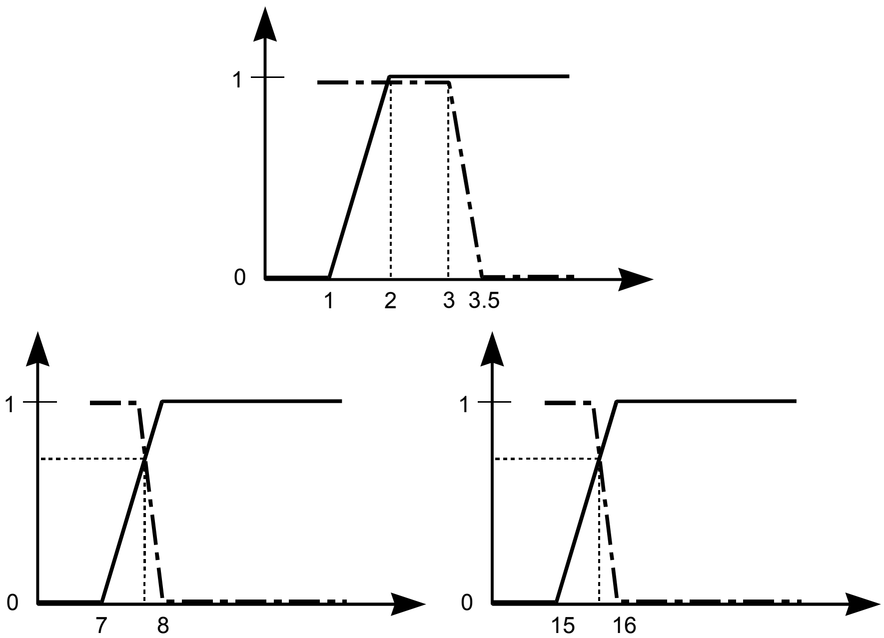

Figure 8. Time windows after constraints propagation

As described previously in the paper, the satisfaction level of least satisfied constraint $[\widetilde{C}_i^L, \widetilde{C}_i^R]$ is a measure of a global satisfaction level. Indeed, in the propagation algorithm there were no new assumptions, so the final, narrower time windows are the only constraints on start times. In [9] Fortemps showed the algorithm which guarantees optimality (in the sense of 'discrimin' order). In general, the algorithm works by iterative repeating the following steps:

1. Apply constraints propagation in order to find global satisfaction level for the solution x, $\alpha^* = \mathrm{Sat}(x)$
2. Every constraint $\widetilde{C}_i$, for which $\forall \alpha > \alpha^* : (\widetilde{C}_i)_\alpha = \emptyset$ (i.e. whose height is equal to $\mathrm{Sat}(x)$; so-called *critical constraint*) is changed to non-fuzzy constraint $C_i' = (\widetilde{C}_i)_{\alpha^*}$
3. Every non-critical constraint is modified to make sure it is satisfied at the level greater than α^*: $\forall \alpha < \alpha^* : (\widetilde{C}_i')_\alpha = (\widetilde{C}_i)_{\alpha^*}$
4. Repeat from step 1., until all constraints are considered critical ones.

The outcome of the above algorithm, the solution with settled satisfaction level α_i^* for every constraint $\widetilde{C}_i$, is optimal in 'discrimin' order.

Continuing previous example, constraints $\widetilde{C}_2$ and $\widetilde{C}_3$ are reduced to singletons, {7.67} and {15.67} respectively. $\widetilde{C}_1$ is modified (step 3) to the form show on figure 9 and possible values are reduced to the [1.67..3.17] interval.

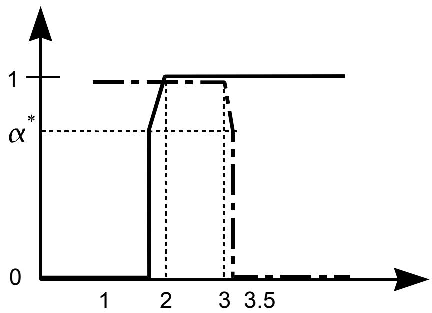

Figure 9. Time window modification of 'discrimin'-optimal solution

6. Uncertain parameters

6.1. Options for uncertainty

As shown in [6], fuzzy quantities can be used to model both flexible constraints and uncertainty. The most popular approach to model uncertainty is probability theory. However, it has very strong assumptions:

- probability theory can be used when representative historical data is available. In contrast, fuzzy sets do not require such assumptions and can be used with little knowledge about historical data They express intuitive knowledge rather than exact uncertainty distribution. Fuzzy number in a sense of Dempster-Shafer theory may be understood as an imprecise probability distribution.
- the operations on probability distribution functions are complex. In contrast, arithmetic on fuzzy sets in simple and naturally extended version of arithmetic on real numbers

Additionally, fuzzy quantities can be interpreted as the family of probability measures ([7]).

For all these reasons, and because fuzzy sets have already been used to model time windows, we will use fuzzy numbers as the representation of uncertain values. Modeling both flexibility and uncertainty by means of fuzzy numbers allows us to propose a coherent approach to solve optimization problems. The variables which are considered uncertain are the travel times between service places. The knowledge about its values is taken from shippers and modeled as 3-points fuzzy numbers. The modal value a^* (one-element core) represents knowledge of the type "travel time is around a^*" and support $[\underline{a}, \overline{a}]$ models statement "travel time lays between $\underline{a}$ and $\overline{a}$" (figure 10).

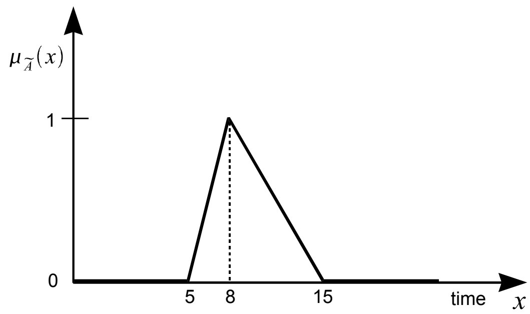

Figure 10. Exemplary fuzzy travel time

Travel times influence:

- time windows fulfillment
- service start times
- objective function

6.2. Satisfying flexible constraints, uncertainly

The uncertainty of travel times naturally is propagated to arrival times and service start times T_i^k. Thus we need to modify equation (54) to incorporate this fuzziness of T_i^k (written from now as $\tilde{T}_i^k$).

Let us split the problem into two cases, one for meeting left part of the constraints $\tilde{C}_i$, second for satisfying right part of constraints. The reason to do so is because the lower bound ($\tilde{C}_i^L$) is a crisp one, either satisfied fully or not at all. On the other hand, the upper constraint ($\tilde{C}_i^R$) has impact on $\mathrm{Sat}(x)$ specified with formula (54). We will start with the upper bound.

Let us shortly recall the interpretation of necessity measure $N(P)$: it describes to what extent we have to believe in $x \in P$, or how much we can doubt about $x \notin P$. If P describes the set of time values not later than service start time t_i ($P=(-\infty, t_i]$), then $N(P)$ is interpreted as the certainty that "by the moment t_i the service has already been started". So $N(P)$ can be seen as the *preference* of such a choice of t_i: the greater the necessity measure value, the more we are sure of the start of service by the time of interest. Using the value of $N(P)$, we can later on define satisfaction level of $\tilde{C}_i^R$ by the value of $\tilde{T}_i$ as the maximal fulfillment level of both $\tilde{C}_i^R$ and $N(P)$ at the same time ([6]):

$$\sup_{t_i} \min\{\mu_{\tilde{C}_i^R}(t_i), N((-\infty, t_i])\} = \sup_{t_i} \min\{\mu_{\tilde{C}_i^R}(t_i), \inf_{t>t_i} \mu_{\tilde{T}_i}(t)\} \quad (69)$$

The figure 11 shows the example of how to determine fulfillment level of $\tilde{C}_i^R$ by fuzzy number $\tilde{T}_i$.

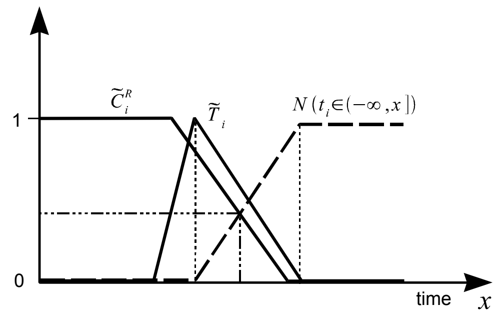

Figure 11. Exemplary right fuzzy window satisfied by fuzzy value

Now, the formula (54) takes form:

$$\mathrm{Sat}(x) = \mu_{\widetilde{C}_1 \cap \ldots \cap \widetilde{C}_{2n}}(x) = \min_{i \in N, k \in K} \mu_{\widetilde{C}_i^R}(\widetilde{T}_i^k) \tag{70}$$

Let $\widetilde{arr}_i^k$ be the arrival time at point *i* by vehicle *k*. Because it occurs after service in previous place on the route of *k*, *j*, it can be calculated as:

$$\widetilde{arr}_i^k = \widetilde{T}_j^k \oplus \tilde{t}_{j,i}^k \tag{71}$$

As we mentioned above, left constraint $\widetilde{C}_i^L$ is either satisfied at level 1 or at level 0. That is why service at place *i* cannot start before neither $\widetilde{C}_i^L$ nor $\widetilde{arr}_i$. Moreover, the membership value of $\mu_{\widetilde{T}_i}(t_i)$ cannot be greater than max($\mu_{\widetilde{C}_i^L}(t_i)$, $\mu_{\widetilde{arr}_i}(t_i)$): there is no possibility to start service before the beginning of $\widetilde{C}_i^L$, even if the vehicle is already there. After beginning of $\widetilde{C}_i^L$ service can be started.

Taking it into consideration, there are four possible relations between $\widetilde{C}_i^L$ and $\widetilde{arr}_i$, and the calculated values of $\widetilde{T}_i$ (see figure 12). Additionally, the dashed line presents approximation made to adjust fuzzy number to 3-points representation. Approximated value is not smaller than the real one.

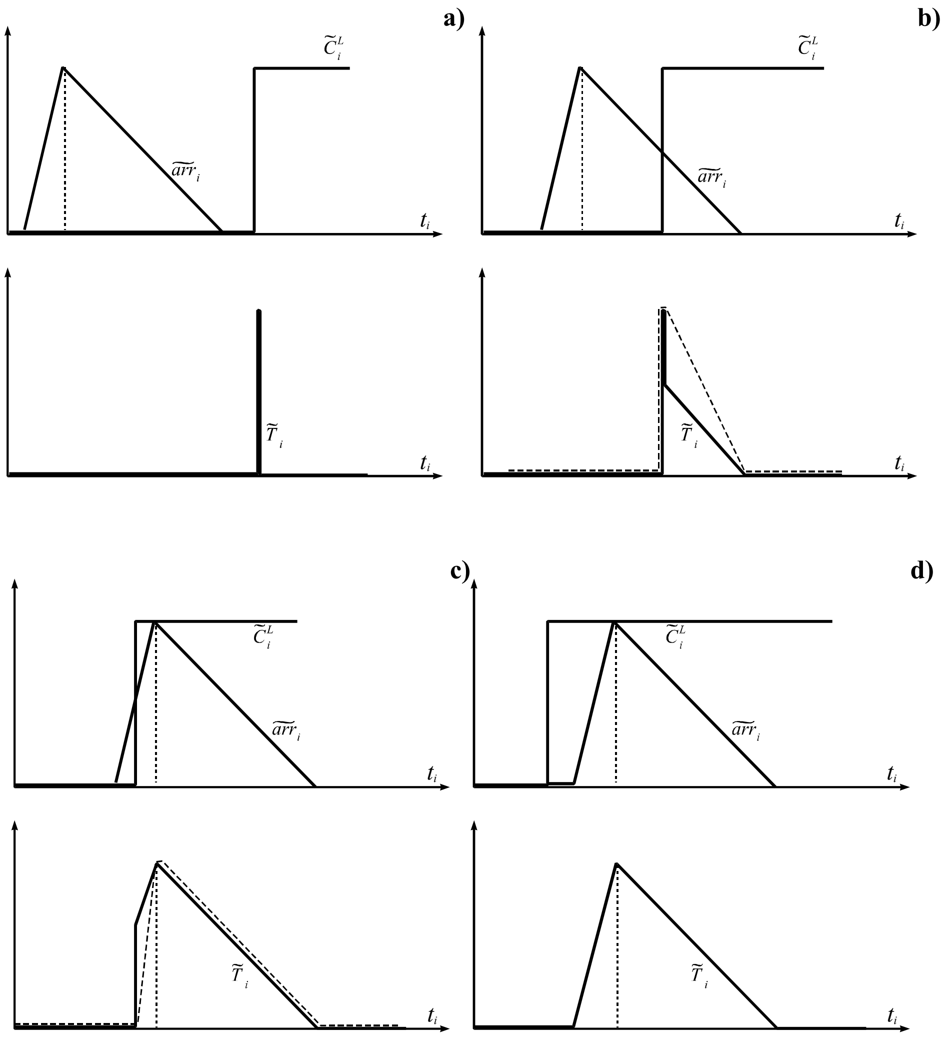

Figure 12. Calculating earliest fuzzy service start time

The cases are following:

- case a): all possible arrival times occur before the beginning of time window
- case b): the possible arrival times occur before as well as after beginning of the time window, but the modal value occurs before the beginning of the time window
- case c): the possible arrival times occur before as well as after the beginning the time window, but the modal value occurs after the beginning of the time window
- case d): all possible arrival times occur after the beginning of the time window

The operation "right" can also be obtained by using following formula:

$$\mathrm{right}(\widetilde{arr}_i, \widetilde{C}_i^L) = \max\{\widetilde{arr}_i, \overline{\widetilde{C}_i^L}\} \quad (72)$$

The interpretation of this formula is that the service must start not earlier than arrival time $\widetilde{arr}_i$ and the set of values describing the time moments before the beginning of time window (hence the fuzzy set complement $\overline{\widetilde{C}_i^L}$).

In the following of the paper, the function "right":

$$\mathrm{right}(\widetilde{T}, \widetilde{C})$$

will describe the number received by using the above procedure of "cutting" the fuzzy number $\widetilde{T}$ by the left fuzzy constraint $\widetilde{C}$ (leaving the right part of $\widetilde{T}$, hence the procedure name).

6.3. Fuzzy objective function

Till now, the objective function (49) has not been concerning the awaiting costs at the service site.

Let $tcost_i^k$ be the awaiting cost for vehicle *k* at place *i* for unit length of time. Starting at the beginning of the route, the vehicle waits in the depot until it is scheduled to drive to the first service place. The awaiting time is $T_{o(k)}^k - a_{o(k)}$ and the awaiting cost is

$$tcost_{o(k)}^k (T_{o(k)}^k - a_{o(k)}) \quad (73)$$

As we can see, this value is crisp, because there are no uncertain travel times, which could influence the start of service time.

Later on the route, in every place *i*, the awaiting time is equal to $\widetilde{T}_i^k - \widetilde{arr}_i^k$ and derived from that, the awaiting cost is

$$tcost_i^k (\widetilde{T}_i^k - \widetilde{arr}_i^k) \quad (74)$$

This value is fuzzy, because of the fuzziness of travel times preceding the place *i* on the route.

Summarizing all costs, the modified objective function (49) has the form:

$$\max \sum_{k \in K} \left(\sum_{i \in N^P} (inc_i \sum_{j \in V^k} X_{ji}^k) - \sum_{(i,j) \in A^k} X_{ij}^k \left(c_{ij}^k + tcost_j^k (\widetilde{T}_j^k - \widetilde{arr}_j^k) \right) - tcost_{o(k)}^k (T_{o(k)}^k - a_{o(k)}) \right) \quad (75)$$

In the formula the costs of awaiting has been incorporated into global costs of travel between places, which is natural, because they *are* generated in the places of service and *are* dependent on travel times.

7. Fuzzy schedule construction

It is assumed, that the routes for all vehicles are established (using some heuristics, for example) and the next step is to fix the schedules. Since in this procedure we assume customer satisfaction more important criterion, first the solution that maximizes the global satisfaction degree is found, then the attempt to increase income (75) is made, without the loss of achieved satisfaction levels.

In the proposed approach the algorithm starts with calculating earliest starting times. Using formula (71), procedure of calculating $\tilde{T}_i^k$ from $\widetilde{arr}_i^k$ and $\tilde{C}_i^L$, and propagation of earliest starting time constraints, we can find *earliest service time*, $\tilde{T}_i^{\inf}$. Surely, the earliest service time in $o(k)$ is

$$\tilde{T}_{o(k)}^{\inf}=(a_{o(k)},a_{o(k)},a_{o(k)}) \qquad \forall k\in K, \tilde{C}_{o(k)}^L=(a_{o(k)},a_{o(k)},\infty,\infty) \tag{76}$$

In the next route places of vehicle *k* the earliest arrival time and earliest service time are

$$\begin{aligned}&\widetilde{arr}_j^{\inf}=\tilde{T}_i^{\inf}\oplus\tilde{t}_{ij}^k\\&\tilde{T}_j^{\inf}=\text{right}(\widetilde{arr}_j^{\inf},\tilde{C}_j^L)\end{aligned} \qquad \forall k\in K, \forall(i,j)\in A^k \tag{77}$$

Then, fulfillment levels of constraints $\tilde{C}_i^R$ are calculated in corresponding time points $\tilde{T}_i^{\inf}$. These levels, marked $\mu_{\tilde{C}_i^R}^*$, are optimal for given routes, because time points are smallest possible and satisfaction functions $\tilde{C}_i^R$ are non-decreasing:

$$\mu_{\tilde{C}_i^R}^*=\mu_{\tilde{C}_i^R}(\tilde{T}_i^{\inf}) \tag{78}$$

It is possible that there are also later moments in time which satisfy all appropriate constraints at the same, optimal levels. Manipulation of these times allows us to decrease awaiting costs.

The *slack* on constraint $\tilde{C}_i^R$ is the set of values of $\tilde{T}_i$ not earlier that $\tilde{T}_i^{\inf}$ and satisfying $\tilde{C}_i^R$ with degree $\mu_{\tilde{C}_i^R}^*$.

Because the service start times depend on arrival times, the uncertainty of the former depend on the uncertainty of the latter. For this reason we assume that the slack contains only times which are constrained by $\tilde{C}_i^L\oplus\Delta t_i$ from the left side (for $\Delta t_i\geq 0$), according to the procedure “right” (figure 13):

$$\tilde{T}_i=\text{right}(\widetilde{arr}_i,\tilde{C}_i^L\oplus\Delta t_i) \tag{79}$$

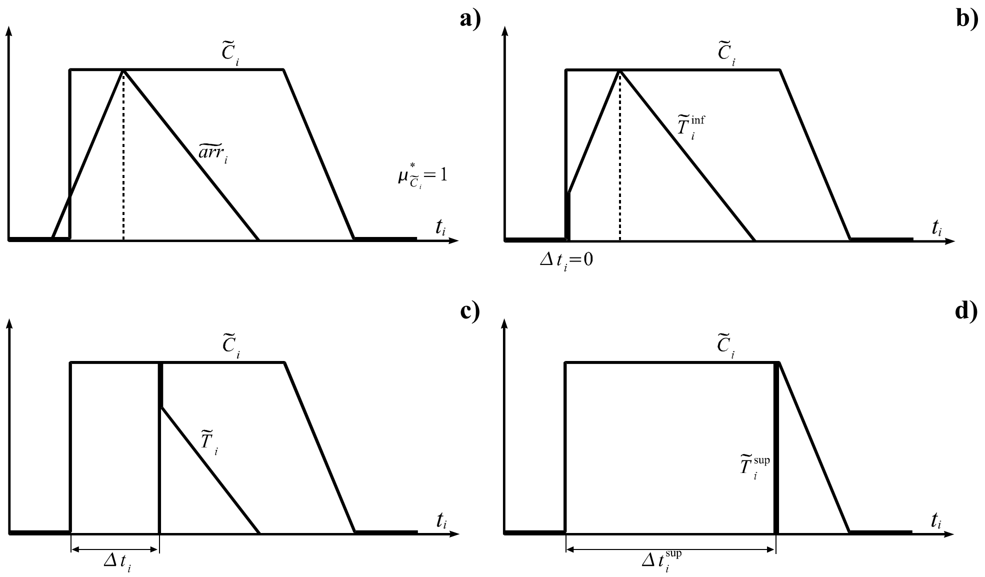

Figure 13. Calculating earliest fuzzy service start time

Such numbers are unambiguously determined by parameter Δt_i, so the slack can alternatively be defined using the value of $\Delta t_i^{\sup}$:

$$\Delta t_i^{\sup}=\max\{\Delta t_i : \mu_{\tilde{C}_i^R}(\mathrm{right}(\widetilde{arr}_i, \tilde{C}_i^L \oplus \Delta t_i))=\mu^*_{\tilde{C}_i^R}\} \tag{80}$$

Backward propagation of constraints (formula (68)) can also be used to calculate the slack of constraints.

The slack of the constraint in the last point on the vehicle route, $\tilde{C}_{d(k)}$, does not need to be modified, because there are no following constraints. For every previous service points, the latest moment of service start cannot be shifted from the earliest more than in the next point the latest moment of service start from the moment of arrival:

$$\Delta t_i^{\sup} \leq \Delta t_j^{\sup}+(\tilde{T}_j^{\inf}-\widetilde{arr}_j^{\inf}) \qquad \forall k \in K, \forall (i,j) \in A^k \tag{81}$$

In other words, the service in i cannot be delayed more than the service in j plus the shift $\tilde{T}_j^{\inf}$ from $\widetilde{arr}_j^k$. In propagation procedure, the following equation is used:

$$\Delta t_i^{\sup}=\min\{\Delta t_i^{\sup}, \Delta t_j^{\sup}+(\tilde{T}_j^{\inf}-\widetilde{arr}_j^{\inf})\} \qquad \forall k \in K, \forall (i,j) \in A^k \tag{82}$$

starting from the end of the route.

Even knowing the earliest starting time and the slack for every service point, it is not generally easy to minimize the summary cost for the whole route. For simplicity, we'll describe the heuristic of minimizing the cost in two consecutive places, i and j .

The cost function for two places of service is non-continuous and depends on the relation between arrival time and service start time. There are two extreme situations, in which the cost can be minimal:

1. $\widetilde{T}_i = \widetilde{T}_i^{\inf}$; the awaiting time in i is minimal, the awaiting time in j is maximal
2. $\widetilde{T}_j = \widetilde{arr}_j^k = \widetilde{T}_j \oplus \tilde{t}_{ij}$; the awaiting time in j is minimal (zero), the awaiting time in i is smallest possible for configurations in which the awaiting time in j is zero

Any other situation leads to greater costs, because they are a linear combination of the two above, or their awaiting times are not smaller than received in case 2. The choice of better situation depends on the unit awaiting cost, $tcost_i^k$: if $tcost_i^k < tcost_j^k$, the case 2. will be chosen. Otherwise, case 1. will be chosen.

8. Summary

In this paper a generalization of vehicle routing problem with time windows has been proposed. The model takes into account flexibility of time constraints and uncertainty of travel times. Both flexibility and uncertainty is modeled by means of fuzzy numbers. This allowed the authors to propose a coherent solution procedure for scheduling vehicles. The procedure yields a schedule that is feasible for every realization of uncertain time parameters. Such an assumption requires a special care of succession of fuzzy time events. The procedure guaranties that the following time events, e.g. beginning of the flexible time window, and fuzzy service starting time, do not occur in reversal order. The procedure is performed in two steps. Since it assumes customer satisfaction more important criterion, first the solution that maximizes the global satisfaction degree is found, then improving procedure tries to increase the profit, such that the achieved satisfaction levels are kept. The first step is reached by propagation of fuzzy constraints and fuzzy travel times in time. During this phase slack times are calculated and then in the second step within the slack times a heuristic searches for such a schedule that is the most profitable.

The authors believe that the generalization of the model has many practical implications. The motivation for introducing new features in the VRP model comes from the observation of real shipper work. Having commercial software systems that implement crisp time windows and crisp travel times in daily work they obtain results and then try to find out what will happen if they relax constraints or the travel time is longer. Application of the proposed model and the solution procedure in commercial software systems would undoubtedly make the life of a shipper easier.

There are other constraints often considered flexible that could be taken into account in the model. Some of them are based on a company processes others the local law. Further research should include them in mathematical model and assuming fuzzy model of such a flexibility propose respective solution procedures.

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