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On the Complexity of Sprite Packing

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Abstract

In this report we provide complexity results on problems related to the construction of CSS-sprites.

1 Max Pictures for Shared Palette

Informally, the problem of the maximum set of pictures for a shared palette consists in selecting as many pictures as possible from a given set such that their colors are covered by the shared palette of a limited size. We will show **NP**-completeness of this problem.

Consider set $\mathcal{I}$ of n images. Image i has palette (i.e. set) of colors p_i from some spectrum U of size $|U| = |\cup_{i=1}^{n} p_i|$. Thus, a palette of size at most $|U|$ is needed to index all colors of $\mathcal{I}$. Given is a limit $l \leq |U|$ on the shared palette size. We formulate our problem as follows:

Max Pictures for Shared Palette
Input: Set $\mathcal{I}$ of images with palettes $p_1, \dots, p_n$, shared palette size l, positive integer m.
Question: Is there a subset $\mathcal{I}' \subseteq \mathcal{I}$ such that $|\cup_{i \in \mathcal{I}'} p_i| \leq l$, and $|\mathcal{I}'| \geq m$, i.e. is it possible to cover at least m pictures from $\mathcal{I}$ by palette of size l?

Theorem 1 *Max Pictures for Shared Palette is* **NP***-complete.*

Proof. Max Set of Pictures for Shared Palette is in **NP** because NDTM can guess set $\mathcal{I}'$ in time $O(m) \leq O(|\mathcal{I}|)$, and verify whether $|\cup_{i \in \mathcal{I}'} p_i| \leq l$ in $O(|\mathcal{I}||U|)$ time.

Next, we show that Balanced Complete Bipartite Subgraph (problem GT24 in [2]) polynomially transforms to our problem. The former problem is defined as follows:

Balanced Complete Bipartite Subgraph (BCBS)
Input: Bipartite graph (V_1, V_2, E), positive integer k.
Question: Are there two disjoint sets $X_1 \subseteq V_1$, $X_2 \subseteq V_2$ such that $|X_1| = |X_2| = k$ and such that $u \in X_1, v \in X_2$ implies $\{u, v\} \in E$.

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Thus, the question in BCBS asks for a biclique $K_{k,k}$. Let $n(i)$ denote neighbors of node $i \in V_1$. In the transformation from BCBS to Max Pictures for Shared Palette nodes of V_1 correspond with the pictures of $\mathcal{I}$ and nodes in V_2 with colors in U. Thus, we have: $|\mathcal{I}| = |V_1|$, $|U| = |V_2|$, $p_i = V_2 \setminus n(i)$, i.e. palette p_i is a complement of the neighbors in $n(i)$. We ask if it is possible to cover $m = k$ pictures with a palette of size $l = |V_2| - k$. The transformation can be done in polynomial time $O(|E|)$.

Suppose the answer to BCBS is positive and the required sets X_1, X_2 exist. We construct set $\mathcal{I}'$ using pictures corresponding to the nodes of X_1. The palette has colors in $V_2 \setminus X_2$ and size $l = |V_2| - k$. Note that picture $i \in \mathcal{I}'$ uses colors in $V_2 \setminus n(i)$ and hence *no* colors from X_2. Since $\forall i \in X_1, j \in X_2, \{i,j\} \in E$ the picture corresponding to i is using no colors from X_2 and a palette of size $|V_2| - k = l$ is sufficient to cover all pictures in $\mathcal{I}'$.

Suppose the answer to Max Pictures for Shared Palette is positive and set $\mathcal{I}'$ of $m = k$ pictures covered by a palette of size $l = |V_2| - k$ exists. It means that $|V_2| - l = k$ colors are not used by any picture in $\mathcal{I}'$ and have been eliminated from the palette. Since picture i uses colors which are complement of $n(i)$, the instance of BCBS has an edge from each node corresponding to a picture in $\mathcal{I}'$ to each node corresponding to the colors absent from the palette. Hence, nodes of X_1 corresponding to $\mathcal{I}'$ and the nodes of X_2 corresponding to unused colors form biclique $K_{k,k}$ and the answer to BCBS is positive. $\square$

2 Picture Alignment

The problem of picture alignment may be formulated as follows: given a set of pictures align them horizontally for maximum overlap of colors on neighboring sides. Picture alignment problem has practical motivation. When packing pictures into a CSS-sprite some pictures will be in direct horizontal contact, i.e. their vertical edges touch each other. If a pair of neighboring pictures have edges of different colors then more data is stored to encode the different neighboring colors, than if the colors were the same. The best alignment of the pictures minimizes the number of color changes. We show that picture alignment problem is **NP**-complete.

More formally picture alignment problem may be formulated as follows. Given is a set $\mathcal{I}$ of n rectangular images. For the sake of conciseness picture features and graphical compression are very simple. Only pixels on the vertical sides of a picture matter for packing efficiency. Therefore, picture i is defined by the sequence of pixel colors l_i on the left side and the sequence of pixel colors r_i on the right side. l_i, r_i are arrays, and pixel x of l_i can be referred to as $l_i[x]$. We assume that all pictures have the same height. If $r_i[y] = l_j[y]$, i.e. y-th right-edge pixel of some picture y is the same as the y-th left-edge pixel of picture j, then the cost of encoding the pair is 1. Otherwise, $r_i[y] \neq l_j[y]$ and the cost of encoding them is equal 2.

Picture Alignment
Input: Set $\mathcal{I}$ of images n of size $2 \times k$ with pixels l_i, r_i, for $i = 1, \dots, n$, on the left and the right side respectively, positive integer l.
Question: Is there a sequence of images such that their cost of packing is not greater than l?

Theorem 2 *Picture alignment is* **NP***-complete.*

Proof. The problem is in **NP** because NDTM can guess the sequence of length n and calculate the cost of packing in time $O(nk)$. Next, we give polynomial time transformation from Hamiltionian Path problem [1, 2]:

Hamiltonian Path (HP)
Input: Graph (V, E).
Question: Is there a Hamiltonian path in G, i.e. a path visiting each node of G once?

In our transformation nodes of HP correspond with pictures. The set of the right pixels serves for identifying nodes while colors in l_i make for edges. Hence, $|\mathcal{I}| = n = |V|$, $k = n$, $r_i[i] = 1, \forall j \neq i, r_i[j] = 2$, for pictures $i = 1, \dots, n$. $\forall \{i, j\} \in E, l_j[i] = 1$ and otherwise $l_j[i] = 0$ (cf. Fig1). We ask whether it is possible to pack the pictures in $\mathcal{I}$ with the cost not greater than $l = (n-1)(2k-1)$. If $\{i, j\} \in E$ then pictures i aligned on the left of j have one neighboring equal pixel and cost of packing j after i is $2k-1$. Otherwise $\{i, j\} \notin E$ and all k pixels are different and the cost of packing j after i is $2k$.

If a Hamiltionian path exists in HP, then we use the sequence of pictures corresponding to the sequence of G nodes. The cost of packing is $l = (n-1)(2k-1)$. If a packing of cost not greater than $(n-1)(2k-1)$ exists then it means that the number of different colors in each aligned pair is $k-1$ because it is not possible to have a smaller difference. This means that between each pair of nodes corresponding to the pair of pictures in the sequence is an edge in G. Hence, Hamiltonian path also exists. $\square$

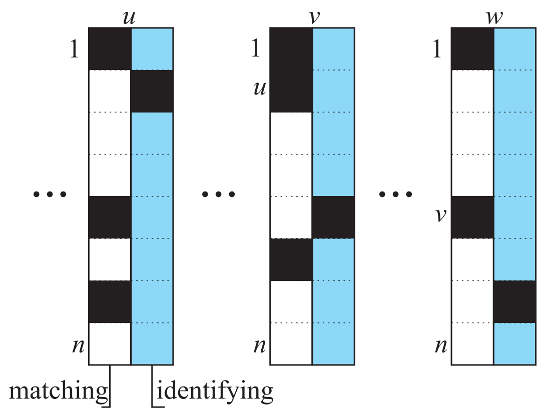

Figure 1: Pictures in the picture alignment problem

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References

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[2] R.Karp, Reducibility Among Combinatorial Problems. In: Miller RE, Thatcher JW, editors. Complexity of Computer Computations, New York, Plenum Press, 1972, p. 85-103.