Finding Similar Items II

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Bachelor studies, eighth semester
Academic year 2018/19 (summer semester)

Review of the previous lectures

- Processing of massive datasets
- Evolution of database systems
- OLTP and OLAP systems
- ETL
- Dimensional modeling
- Data processing
- MapReduce in Spark
- Approximate query processing
- Finding similar items:
  - Minhash signatures

Outline

1. Locality-Sensitive Hashing for Documents
2. Distance measures
3. Theory of Locality-Sensitive Functions
4. LSH Families for Other Distance Measures
5. Summary

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  - The entire data fits in a gigabyte – less than a typical main memory of a laptop.
  - However, there are $\binom{1000000}{2}$ or half a trillion pairs of documents.
  - If it takes a microsecond to compute the similarity of two signatures, then it takes almost six days to compute all the similarities on that laptop.

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- A technique called **locality-sensitive hashing** (**LSH**) is a solution for this problem.

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- We hope to have a small fraction of false positives and false negatives.

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band 1	$\cdots$	3	2	1	2	2	$\cdots$
	$\cdots$	0	1	3	1	1	$\cdots$
	$\cdots$	5	3	5	1	3	$\cdots$
band 2	$\cdots$	1	4	1	2	4	$\cdots$
	$\cdots$	6	1	6	1	1	$\cdots$
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- We assume that the chances of an accidental collision to be very small.

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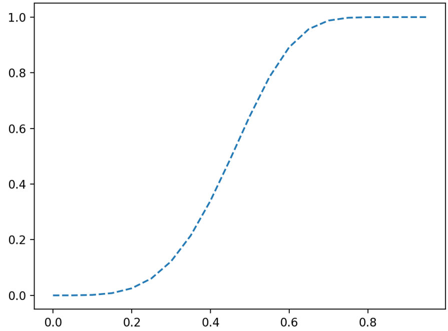
- The probability that two documents become a candidate pair has a form of an *S*-curve.




```
>>> import numpy as np
>>> import matplotlib.pyplot as plt
>>> s = np.arange(0., 1., 0.05)
>>> plt.plot(s, 1-(1-s**4)**16, '--')
>>> plt.show()
```

Analysis of the banding technique

- The threshold, the value of similarity s at which the rise becomes steepest, is a function of b and r.
- Use `sympy` to compute the threshold:

```
>>> from sympy import *
>>> s,r,b=Symbol('s'),Symbol('r'),Symbol('b')
>>> d = diff(1-(1-s**r)**b, s, 2)
>>> solve(d,s)
[((r - 1)/(b*r - 1))**(1/r)]
```

- An approximation to the threshold is $(1/b)^{1/r}$.
- **Example**: for $b = 16$ and $r = 4$, the threshold is approximately $1/2$.

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    - However, there are 20 bands and thus 20 chances to become a candidate.
    - That is why the final probability is 0.99965 (since the probability of a false negative is 0.00035).

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- If avoidance of false negatives is important, you may wish to select b and r to produce a threshold lower than t.
- If speed is important and you wish to limit false positives, select b and r to produce a higher threshold.

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  4. $d(\boldsymbol{x}, \boldsymbol{y}) \leq d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y})$ (the triangle inequality).
- The triangle-inequality axiom is what makes all distance measures behave as if distance describes the length of a shortest path from one point to another.

Euclidean distances

- The conventional distance measure in n-dimensional Euclidean space, which we shall refer to as the L_2-norm, is defined as:

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

- In general, for any constant p, we can define the L_p-norm to be the distance measure d defined by:

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{\frac{1}{p}}$$

Euclidean distances

- Special cases are, besides the L_2-norm mentioned above,
  - Manhattan distance or L_1-norm:

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \left(\sum_{j=1}^{n} |x_j - y_j| \right)$$

  - Chebyshev distance or L_∞-norm:

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_j (|x_j - y_j|)$$

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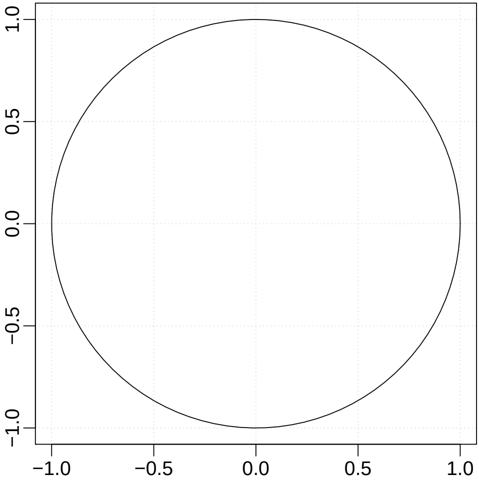

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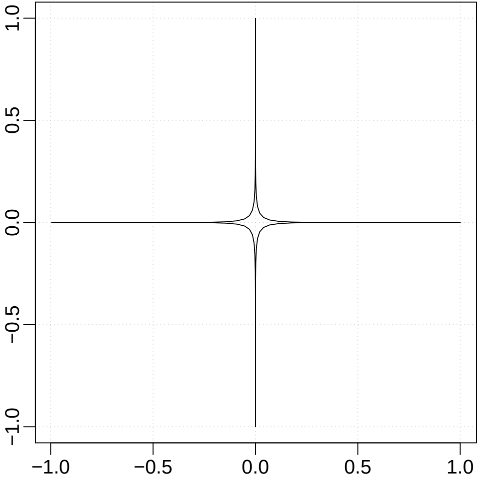

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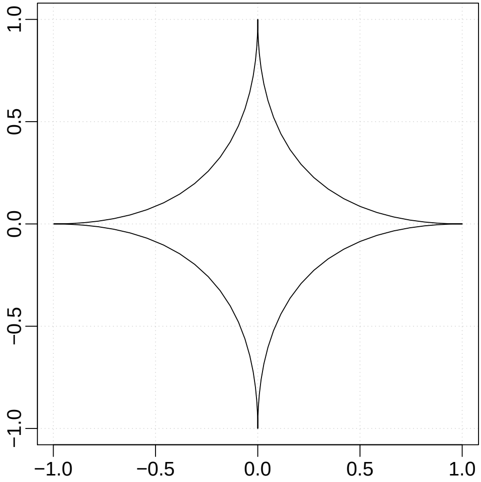

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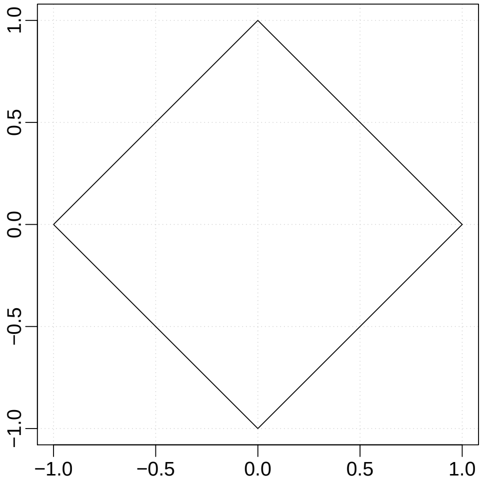

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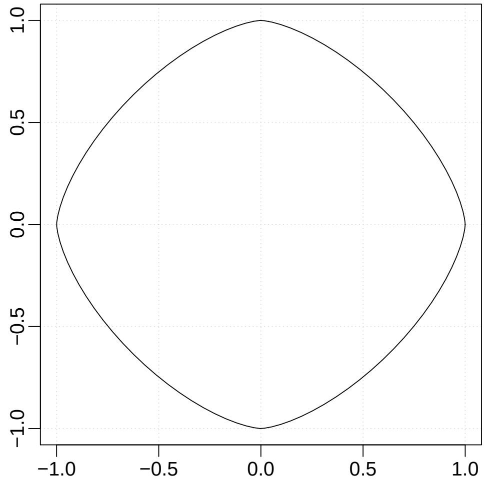

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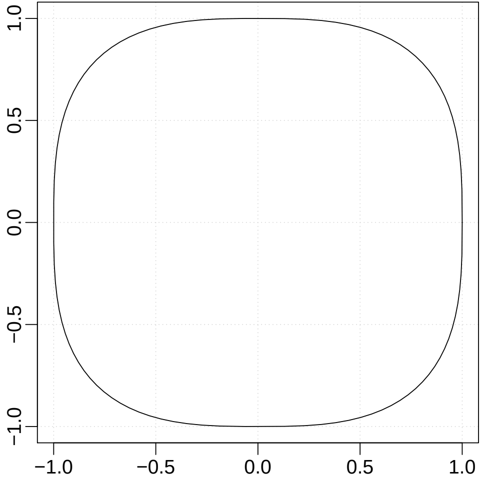

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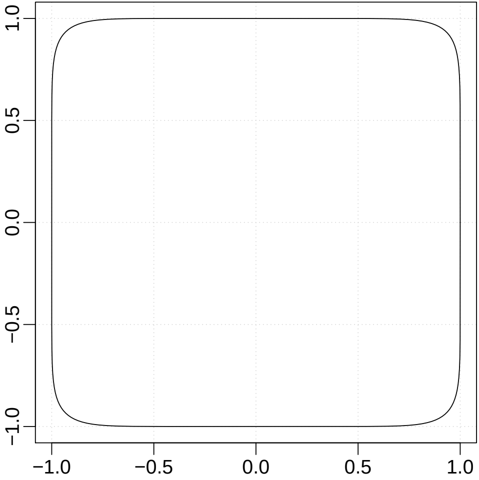


Jaccard distance

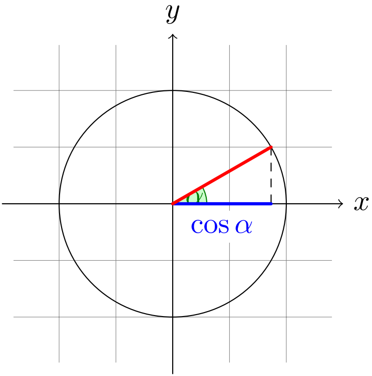
- Jaccard similarity is not a distance measure!
- We define the Jaccard distance of sets by:

$$d_{Jacc} = 1 - SIM(\boldsymbol{x}, \boldsymbol{y})$$

where $SIM(\boldsymbol{x}, \boldsymbol{y})$ is defined as before.

Cosine distance

- Let points be thought of as directions and do not distinguish between a vector and a multiple of that vector.



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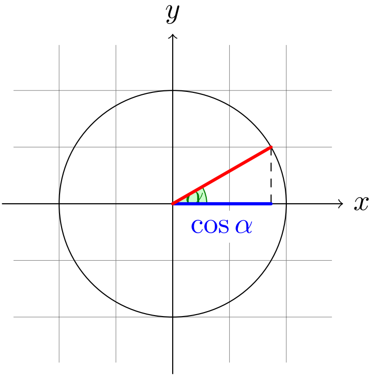
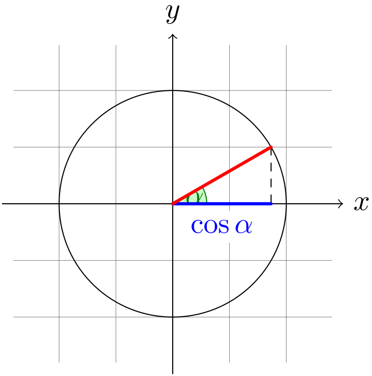
- The cosine distance between two points is the angle that the vectors to those points make.



Cosine distance

- Let points be thought of as directions and do not distinguish between a vector and a multiple of that vector.
- The cosine distance between two points is the angle that the vectors to those points make.
- This angle will be in the range 0 to 180 degrees, regardless of how many dimensions the space has.



Computing the cosine distance

- Given two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, the cosine of the angle between them is the dot product $\boldsymbol{x} \cdot \boldsymbol{y}$ divided by the L_2-norms of $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$cos(\theta) = \frac{\sum_{j=1}^{n} x_j y_j}{\sqrt{\sum_{j=1}^{n} x_j^2}\sqrt{\sum_{j=1}^{n} y_j^2}}$$

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- Apply the $\arccos$ function to translate $\cos(\theta)$ to an angle in the $[0, 180]$ degree range.

Hamming Distance

- The Hamming distance between two vectors is the number of components in which they differ:

$$d_H = \sum_{j=1}^{n} [\![x_j \neq y_j]\!]$$

Outline

Theory of locality-sensitive functions

- For a given distance measure we would like to find a family of functions that can be combined to distinguish strongly between pairs at a low distance from pairs at a high distance.

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- The minhash functions is one example of such family that uses the banding technique to achieve the above goal.

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     - They must be combinable to build functions that are better at avoiding false positives and negatives, and the combined functions must also take time that is much less than the number of pairs.

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- A collection of functions of this form will be called a family of functions.

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  - For instance, for $d_1 = 0.3$ and $d_2 = 0.6$ we can assert that the family of minhash functions is a $(0.3, 0.6, 0.7, 0.4)$-sensitive family.

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- **Example**: This construction corresponds to r rows in a single band for minhash functions.

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- This construction turns a (d_1, d_2, p_1, p_2)-sensitive family $\mathcal{F}$ into a $(d_1, d_2, 1-(1-p_1)^b, 1-(1-p_2)^b)$-sensitive family $\mathcal{F}'$.

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- Obviously, the better the final family of functions is, the longer it takes to apply the functions from this family.

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  - The members of F_2 each are built from 16 members of $\mathcal{F}$.
  - The 4-way AND-function converts any probability p into p^4, and the 4-way OR-construction, converts this probability further into $1 - (1 - p^4)^4$.

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- **Example**:
  - Suppose $\mathcal{F}$ is the minhash functions being a $(0.2, 0.6, 0.8, 0.4)$-sensitive family.

p	$1-(1-p^4)^4$
0.2	0.0064
0.3	0.0320
0.4	0.0985
0.5	0.2275
0.6	0.4260
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  - This family corresponds to the banding technique with $b = 4$ bands and $r = 4$ rows of the banding technique.
  - By replacing $\mathcal{F}$ by $\mathcal{F}_2$, we have reduced both the false-negative and false-positive rates, at the cost of making application of the functions take 16 times as long.

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- **Example**:
  - For the same cost, we can apply a 4-way OR-construction followed by a 4-way AND-construction.
  - Suppose as before that $\mathcal{F}$ is a $(0.2, 0.6, 0.8, 0.4)$-sensitive family.
  - Then the constructed family is a $(0.2, 0.6, 0.9936, 0.5740)$-sensitive.
  - This choice is not necessarily the best: the higher probability has moved much closer to 1, but the lower probability has also raised, increasing the number of false positives.

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- It would, for instance, transform a $(0.2, 0.8, 0.8, 0.2)$-sensitive family into a $(0.2, 0.8, 0.99999996, 0.0008715)$-sensitive family.

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  i.e., the fraction of positions in which $\boldsymbol{x}$ and $\boldsymbol{y}$ agree.
- The family $\mathcal{F}$ consisting of the functions $\{f_1, f_2, \ldots, f_n\}$ is a $(d_1, d_2, 1 - d_1/n, 1 - d_2/n)$-sensitive family of hash functions, for any $d_1 < d_2$.

Random hyperplanes and the cosine distance

- The cosine distance between two vectors is the angle between the vectors.
- Note that these vectors may be in a space of many dimensions, but they always define a plane, and the angle between them is measured in this plane.

Random hyperplanes and the cosine distance

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- To pick a random hyperplane, we may pick the normal vector $\boldsymbol{v}$.

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- Then $\mathcal{F}$ is a $(d_1, d_2, (180 - d_1)/180, (180 - d_2)/180)$-sensitive family for the cosine distance.

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- The segments of the line are the buckets into which function f hashes points: a point is hashed to the bucket in which its projection onto the line lies.

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- For $\theta = 90$ degrees the two points are certain to fall in the same bucket.

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  - Since θ is the smaller angle between two randomly chosen lines in the plane, θ is twice as likely to be between 0 and 60 as it is to be between 60 and 90.

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- For distances at least $2a$ the probability points at that distance will fall in the same bucket is at most $1/3$.
- We can amplify this family as we like, just as for the other examples of locality-sensitive hash functions.

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- Given that $d_1 < d_2$, we may not know what exactly the probabilities of p_1 and p_2 are, but we can be certain that $p_1 > p_2$.
- The reason is that this probability surely grows as the distance shrinks.
- Thus, even if we cannot calculate p_1 and p_2 easily, we know that there is a (d_1, d_2, p_1, p_2)-sensitive family of hash functions for any $d_1 < d_2$ and any given number of dimensions.

Outline

Summary

- Locality-sensitive hashing.
- Distance measures.
- Theory of LSH.
- LSH for different distance measures.

Bibliography

- J. Leskovec, A. Rajaraman, and J. D. Ullman. *Mining of Massive Datasets*. Cambridge University Press, 2014
  `http://www.mmds.org`

- P. Indyk. Algorithms for nearest neighbor search