Finding Similar Items II

Krzysztof Dembczyński

Intelligent Decision Support Systems Laboratory (IDSS)
Poznań University of Technology, Poland

Software Development Technologies
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Review of the previous lectures

• Mining of massive datasets.
• Classification and regression.
• Evolution of database systems.
• MapReduce
• MapReduce in Apache Spark
• Nearest neighbor search:
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• Nearest neighbor search:
  ▶ Minhash technique
  ▶ Locality-sensitive hashing with minhash
Outline

1. Distance measures
2. Theory of Locality-Sensitive Functions
3. LSH Families for Other Distance Measures
4. Summary
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Distance measure

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• A distance measure on this space is a function $d(x, y)$ that takes two points in the space as arguments and produces a real number, and satisfies the following axioms:

1. $d(x, y) \geq 0$ (no negative distances).
2. $d(x, y) = 0$ if and only if $x = y$ (distances are positive, except for the distance from a point to itself).
3. $d(x, y) = d(y, x)$ (distance is symmetric).
4. $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality).

The triangle-inequality axiom is what makes all distance measures behave as if distance describes the length of a shortest path from one point to another.
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• The triangle-inequality axiom is what makes all distance measures behave as if distance describes the length of a shortest path from one point to another.
Euclidean distances

• The conventional distance measure in $n$-dimensional Euclidean space, which we shall refer to as the $L_2$-norm, is defined as:

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

• In general, for any constant $p$, we can define the $L_p$-norm to be the distance measure $d$ defined by:

$$d_p(\mathbf{x}, \mathbf{y}) = \left( \sum_{j=1}^{n} |x_j - y_j|^p \right)^{\frac{1}{p}}$$
Euclidean distances

• Special cases are, besides the $L_2$-norm mentioned above,
  ▶ Manhattan distance or $L_1$-norm:

$$d_1(\mathbf{x}, \mathbf{y}) = \left( \sum_{j=1}^{n} |x_j - y_j| \right)$$

▶ Chebyshev distance or $L_\infty$-norm:

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_j (|x_j - y_j|)$$
Euclidean distances

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• Jaccard similarity is not a distance measure!
• We define the Jaccard distance of sets by:

\[ d_{Jacc} = 1 - SIM(x, y) \]

where \( SIM(x, y) \) is defined as before.
Cosine distance

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• The cosine distance between two points is the angle that the vectors to those points make.
• This angle will be in the range 0 to 180 degrees, regardless of how many dimensions the space has.
Computing the cosine distance

- Given two vectors $x$ and $y$, the cosine of the angle between them is the dot product $x \cdot y$ divided by the $L_2$-norms of $x$ and $y$:

$$
cos(\theta) = \frac{\sum_{j=1}^{n} x_j y_j}{\sqrt{\sum_{j=1}^{n} x_j^2} \sqrt{\sum_{j=1}^{n} y_j^2}}
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- Apply the \( \arccos \) function to translate \( \cos(\theta) \) to an angle in the \([0, 180]\) degree range.
The Hamming distance between two vectors is the number of components in which they differ:

\[ d_H = \sum_{j=1}^{n} \left[ x_j \neq y_j \right] \]
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• For a given distance measure we would like to find a family of functions that can be combined to distinguish strongly between pairs at a low distance from pairs at a high distance.
Theory of locality-sensitive functions

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• The minhash functions is one example of such family that uses the banding technique to achieve the above goal.
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They must be efficient, in two ways:

- They must be able to identify candidate pairs in time much less than the time it takes to look at all pairs.
- They must be combinable to build functions that are better at avoiding false positives and negatives, and the combined functions must also take time that is much less than the number of pairs.
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• A collection of functions of this form will be called a family of functions.
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[Localitätssensitive Funktionen]
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  1. If $d(x, y) \leq d_1$, then the probability that $f(x) = f(y)$ is at least $p_1$.
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Example: For Jaccard distance we have:

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  ▶ Thus, the family of minhash functions is a $(d_1, d_2, 1 - d_1, 1 - d_2)$-sensitive family for any $d_1$ and $d_2$, where $0 \leq d_1 < d_2 \leq 1$.
  ▶ For instance, for $d_1 = 0.3$ and $d_2 = 0.6$ we can assert that the family of minhash functions is a $(0.3, 0.6, 0.7, 0.4)$-sensitive family.
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- **Example**: This construction corresponds to \(r\) rows in a single band for minhash functions.
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This construction turns a $(d_1, d_2, p_1, p_2)$-sensitive family $\mathcal{F}$ into a $(d_1, d_2, 1 - (1 - p_1)^b, 1 - (1 - p_2)^b)$-sensitive family $\mathcal{F}'$. 

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• **Example**: This construction corresponds to $b$ bands of 1 row for minhash functions.
• The AND-construction lowers all probabilities, while the OR-construction makes all probabilities rise.

• But if we choose $F$ and $r$ judiciously, we can make the small probability $p_2$ get very close to 0, while the higher probability $p_1$ stays significantly away from 0.

• Similarly, by choosing $F$ and $b$ judiciously, we can make the larger probability approach 1 while the smaller probability remains bounded away from 1.

• We can, moreover, cascade AND- and OR-constructions in any order to make the low probability close to 0 and the high probability close to 1!!!

• Obviously, the better the final family of functions is, the longer it takes to apply the functions from this family.
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• Example:

Suppose we start with a family $F$. We use the AND-construction with $r = 4$ to produce a family $F_1$. We then apply the OR-construction to $F_1$ with $b = 4$ to produce a third family $F_2$. The members of $F_2$ each are built from 16 members of $F$. The 4-way AND-function converts any probability $p$ into $p^4$, and the 4-way OR-construction, converts this probability further into $1 - (1 - p^4)^4$. 
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  - We use the AND-construction with $r = 4$ to produce a family $\mathcal{F}_1$. 
    - The members of $\mathcal{F}_2$ each are built from $16$ members of $\mathcal{F}$.
    - The 4-way AND-function converts any probability $p$ into $p^4$, and the 4-way OR-construction, converts this probability further into $1 - (1 - p^4)^4$. 

Amplifying a locality-sensitive family

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Amplifying a locality-sensitive family

- **Example:**

  ▶ Suppose $\mathcal{F}$ is the minhash functions being a $(0.2, 0.6, 0.8, 0.4)$-sensitive family.

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  - This family corresponds to the banding technique with $b = 4$ bands and $r = 4$ rows of the banding technique.

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- By replacing \( F \) by \( F_2 \), we have reduced both the false-negative and false-positive rates, at the cost of making application of the functions take 16 times as long.

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Amplifying a locality-sensitive family

- **Example:**
  - For the same cost, we can apply a 4-way OR-construction followed by a 4-way AND-construction.
  - Suppose as before that $\mathcal{F}$ is a $(0.2, 0.6, 0.8, 0.4)$-sensitive family.
  - Then the constructed family is a $(0.2, 0.6, 0.9936, 0.5740)$-sensitive.
  - This choice is not necessarily the best: the higher probability has moved much closer to 1, but the lower probability has also raised, increasing the number of false positives.
Amplifying a locality-sensitive family

• We can cascade constructions as much as we like.
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It would, for instance, transform a $(0.2, 0.8, 0.8, 0.2)$-sensitive family into a $(0.2, 0.8, 0.99999996, 0.0008715)$-sensitive family.
Outline

1. Distance measures
2. Theory of Locality-Sensitive Functions
3. LSH Families for Other Distance Measures
4. Summary
LSH families for Hamming distance

• Suppose we have a space of $n$-dimensional vectors, and $h(x, y)$ denotes the Hamming distance between vectors $x$ and $y$. 

• Take any position $i$ of the vectors and define the function $f_i(x)$ to be the $i$-th element of vector $x$.

• Then $f_i(x) = f_i(y)$ if and only if vectors $x$ and $y$ agree in the $i$-th position.

• The probability that $f_i(x) = f_i(y)$ for a randomly chosen $i$ is: $1 - h(x, y) / n$, i.e., the fraction of positions in which $x$ and $y$ agree.

• The family $F$ consisting of the functions $\{f_1, f_2, ..., f_n\}$ is a $(d_1, d_2, 1 - d_1/n, 1 - d_2/n)$-sensitive family of hash functions, for any $d_1 < d_2$. 

LSH families for Hamming distance

- Suppose we have a space of \( n \)-dimensional vectors, and \( h(x, y) \) denotes the Hamming distance between vectors \( x \) and \( y \).
- Take any position \( i \) of the vectors and define the function \( f_i(x) \) to be the \( i \)-th element of vector \( x \).
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Random hyperplanes and the cosine distance

- The cosine distance between two vectors is the angle between the vectors.
- Note that these vectors may be in a space of many dimensions, but they always define a plane, and the angle between them is measured in this plane.
Random hyperplanes and the cosine distance

• Let the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ be $\theta$. 
Random hyperplanes and the cosine distance

- Let the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ be $\theta$.
- Suppose we pick a hyperplane through the origin of the space that intersects the plane of $\mathbf{x}$ and $\mathbf{y}$ in a line.
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• The hyperplane is the set of points whose dot product with $v$ is 0.
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Take the dot products of $v$ with $x$ and $y$:

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and check the signs of these products.
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$$\theta/180$$
Random hyperplanes and the cosine distance

• Thus, each hash function $f$ in our locality-sensitive family $\mathcal{F}$ is built from a randomly chosen vector $\mathbf{v}_f$. 
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• Then $\mathcal{F}$ is a $(d_1, d_2, (180 - d_1)/180, (180 - d_2)/180)$-sensitive family for the cosine distance.
Consider first a 2-dimensional Euclidean space.
LSH families for Euclidean distance

• Consider first a 2-dimensional Euclidean space.
• Each hash function \( f \) in our family \( \mathcal{F} \) will be associated with a randomly chosen line in this space.
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• Pick a constant $a$ and divide the line into segments of length $a$. 
• Consider first a 2-dimensional Euclidean space.
• Each hash function $f$ in our family $\mathcal{F}$ will be associated with a randomly chosen line in this space.
• Pick a constant $a$ and divide the line into segments of length $a$.
• The segments of the line are the buckets into which function $f$ hashes points: a point is hashed to the bucket in which its projection onto the line lies.
LSH families for Euclidean distance

• If the distance $d$ between two points is small compared with $a$, then there is a good chance the two points hash to the same bucket.
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- If the distance $d$ between two points is small compared with $a$, then there is a good chance the two points hash to the same bucket.
- For $d = a/2$ there is at least a $50\%$ chance the two points will fall in the same bucket.
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• If the distance $d$ between two points is small compared with $a$, then there is a good chance the two points hash to the same bucket.
• For $d = a/2$ there is at least a 50% chance the two points will fall in the same bucket.
• If the angle $\theta$ between the randomly chosen line and the line connecting the points is large, then there is an even greater chance that the two points will fall in the same bucket.
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- If the angle $\theta$ between the randomly chosen line and the line connecting the points is large, then there is an even greater chance that the two points will fall in the same bucket.
- For $\theta = 90$ degrees the two points are certain to fall in the same bucket.
Suppose $d$ is larger than $a$.

- To have any chance of the two points falling in the same bucket, we need $d \cos \theta < a$.

- Note, however, that even if $d \cos \theta \ll a$, it is still not certain that the two points will fall in the same bucket.

- However, we can guarantee that if $d \geq 2a$, then there is no more than $\frac{1}{3}$ chance the two points fall in the same bucket.

- Why? The reason is that for $\cos \theta < \frac{1}{2}$ we have $\theta \in (60, 90)$ degrees, and for $\cos \theta \geq \frac{1}{2}$, we have $\theta \in [0, 60]$ degrees.

- Since $\theta$ is the smaller angle between two randomly chosen lines in the plane, $\theta$ is twice as likely to be between $0$ and $60$ as it is to be between $60$ and $90$. 
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• There are, however, two problems with this family of hash functions:

  ▶ The above reasoning was given only for 2-dimensional spaces.

  ▶ This locality-sensitive family for any pair of distances $d_1$ and $d_2$ needs the stronger condition $d_1 < 4d_2$ than the families before, for which we have $d_1 < d_2$.

• It turns out that there is a locality-sensitive family of hash functions for any $d_1 < d_2$ and for any number of dimensions constructed in a similar way.

• Given that $d_1 < d_2$, we may not know what exactly the probabilities of $p_1$ and $p_2$ are, but we can be certain that $p_1 > p_2$.

• The reason is that this probability surely grows as the distance shrinks.

• Thus, even if we cannot calculate $p_1$ and $p_2$ easily, we know that there is a $(d_1,d_2,p_1,p_2)$-sensitive family of hash functions for any $d_1 < d_2$ and any given number of dimensions.
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• Locality-sensitive hashing.
• Distance measures.
• Theory of LSH.
• LSH for different distance measures.
  http://www.mmds.org

• P. Indyk. Algorithms for nearest neighbor search