Finding similar items I

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Software Development Technologies
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Review of the previous lectures

- Mining of massive datasets.
- Classification and regression.
- Evolution of database systems.
- MapReduce
- MapReduce in Apache Spark

Outline

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Nearest neighbor search

- Find similar elements to the query element.

Applications of nearest neighbor search

- Similarity of documents
  - Plagiarism
  - Mirror pages
  - Articles from the same source
- Machine learning
  - k-nearest neighbors
  - Collaborative filtering
- Computational geometry
- Computer vision
- Geographic Information Systems (GIS)

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- Approximate algorithms

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- We can represents pages as sets of character k-grams (or k-shingles) and formulate a problem as finding sets with a relatively large intersection.
- Storing large number of sets and computing their similarity in naive way is not sufficient.
- We compress sets in a way that enables to deduce the similarity of the underlying sets from their compressed versions.

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- **Example**: Let $S = \{a, b, c, d\}$ and $T = \{c, d, e, f\}$, then

$$SIM(S,T) = 2/6.$$

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- **Example**: Let us check two words *document* and *monument*:

$$\begin{aligned}
SIM(\{d,o,c,u,m,e,n,t\},\{m,o,n,u,m,e,n,t\}) &= 6/8 \\
SIM(\{doc,ocu,cum,ume,men,ent\}, \\
\{mon,onu,num,ume,men,ent\}) &= 3/9
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  - Since typical email is much smaller than 14 million characters long, this can be right value.
  - Since distribution of characters is not uniform, the above estimate should be corrected, for example, by assuming that there are only 20 characters.

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  - Each 9-shingle from a document can be mapped to a bucket number in the range from 0 to $2^{32} - 1$.
  - Instead of **nine** we use then **four** bytes and can manipulate (hashed) shingles by single-word machine operations.

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  - if we use 9-shingles, there are many more than 2^{32} likely shingles.
  - When we hash them down to four bytes, we can expect almost any sequence of four bytes to be possible.

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Similarity-preserving summaries of sets

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- We would like to replace large sets by much smaller representations called **signatures**.
- The signatures, however, should preserve (at least to some extent) the similarity between sets.

Matrix representation of sets

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  - The columns of the matrix correspond to the sets.
  - The rows correspond to elements of the universal set from which elements of the sets are drawn.
  - There is a 1 in row r and column c if the element for row r is a member of the set for column c.
  - Otherwise the value in position (r, c) is 0.

Matrix representation of sets

- **Example**:
  - Let the universal set be $\{a, b, c, d, e\}$.
  - Let $S_1 = \{a, d\}$, $S_2 = \{c\}$, $S_3 = \{b, d, e\}$, $S_4 = \{a, c, d\}$.

Element	S_1	S_2	S_3	S_4
a	1	0	0	1
b	0	0	1	0
c	0	1	0	1
d	1	0	1	1
e	0	0	1	0

- It is important to remember that the characteristic matrix is unlikely to be the way the data is stored, but it is useful as a way to visualize the data!

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- The index of the first row is 0 in the following.

Minhashing

- **Example**:
  - Let us pick the order of rows $beadc$ for the matrix from the previous example.

Element	S_1	S_2	S_3	S_4
b	0	0	1	0
e	0	0	1	0
a	1	0	0	1
d	1	0	1	1
c	0	1	0	1

  - In this matrix, we can read off the values of minhash (mh) by scanning from the top until we come to a 1.
  - Thus, we see that $mh(S_1) = 2$ (a), $mh(S_2) = 4$ (c), $mh(S_3) = 0$ (b), and $mh(S_4) = 2$ (a).

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  - The probability that the minhash function for a random permutation of rows produces the same value for two sets equals the Jaccard similarity of those sets.

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- The signature matrix has the same number of columns as M, but only n rows!
- Even if M is not represented explicitly (but as a sparse matrix by the location of its ones), it is normal for the signature matrix to be much **smaller** than M.

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- Fortunately, it is possible to simulate the effect of a random permutation by a **random hash function** that maps row numbers to as many buckets as there are rows.

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- We can maintain the fiction that our hash function h **permutes** row r to position $h(r)$ in the permuted order.

Computing minhash signatures

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- Let $SIG(i, c)$ be the element of the signature matrix for the i-th hash function and column c defined by

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  - Let us consider two hash functions h_1 and h_2:

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1	0	0	1	0		
2	0	1	0	1		
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2	0	1	0	1	3	2
3	1	0	1	1	4	0
4	0	0	1	0	0	3

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  - However, there are $\binom{1000000}{2}$ or half a trillion pairs of documents.
  - If it takes a microsecond to compute the similarity of two signatures, then it takes almost six days to compute all the similarities on that laptop.

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- If so, then we need to focus our attention only on pairs that are likely to be similar, without investigating every pair.
- A technique called **locality-sensitive hashing** (**LSH**) is a solution for this problem.

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  - Hash items several times, in such a way that similar items are more likely to be hashed to the same bucket than dissimilar items are.

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- We hope to have a small fraction of false positives and false negatives.

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	$\cdots$	1	0	0	0	2	$\cdots$
band 1	$\cdots$	3	2	1	2	2	$\cdots$
	$\cdots$	0	1	3	1	1	$\cdots$
	$\cdots$	5	3	5	1	3	$\cdots$
band 2	$\cdots$	1	4	1	2	4	$\cdots$
	$\cdots$	6	1	6	1	1	$\cdots$
	$\cdots$	3	1	4	6	6	$\cdots$
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	···	3	1	4	6	6	···
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- We assume that the chances of an accidental collision to be very small.

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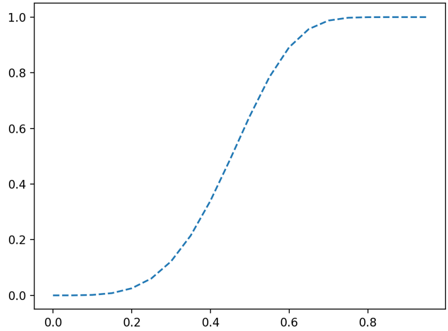
- The probability that two documents become a candidate pair has a form of an S-curve.




```
>>> import numpy as np
>>> import matplotlib.pyplot as plt
>>> s = np.arange(0., 1., 0.05)
>>> plt.plot(s, 1-(1-s**4)**16, '--')
>>> plt.show()
```

Analysis of the banding technique

- The threshold, the value of similarity s at which the rise becomes steepest, is a function of b and r.
- Use `sympy` to compute the threshold:

```
>>> from sympy import *
>>> s,r,b=Symbol('s'),Symbol('r'),Symbol('b')
>>> d = diff(1-(1-s**r)**b, s, 2)
>>> solve(d,s)
[((r - 1)/(b*r - 1))**(1/r)]
```

- An approximation to the threshold is $(1/b)^{1/r}$.
- **Example**: for $b = 16$ and $r = 4$, the threshold is approximately $1/2$.

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    - If we consider two documents with similarity 0.8, then in any one band, they have only about 33% chance of becoming a candidate pair.
    - However, there are 20 bands and thus 20 chances to become a candidate.
    - That is why the final probability is 0.99965 (since the probability of a false negative is 0.00035).

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- If speed is important and you wish to limit false positives, select b and r to produce a higher threshold.

Outline

Summary

- Similarity of documents.
- Jaccard similarity.
- Minhash technique.
- Locality-Sensitive Hashing for Documents.

Bibliography

- J. Leskovec, A. Rajaraman, and J. D. Ullman. *Mining of Massive Datasets*. Cambridge University Press, 2014
- P. Indyk. Algorithms for nearest neighbor search