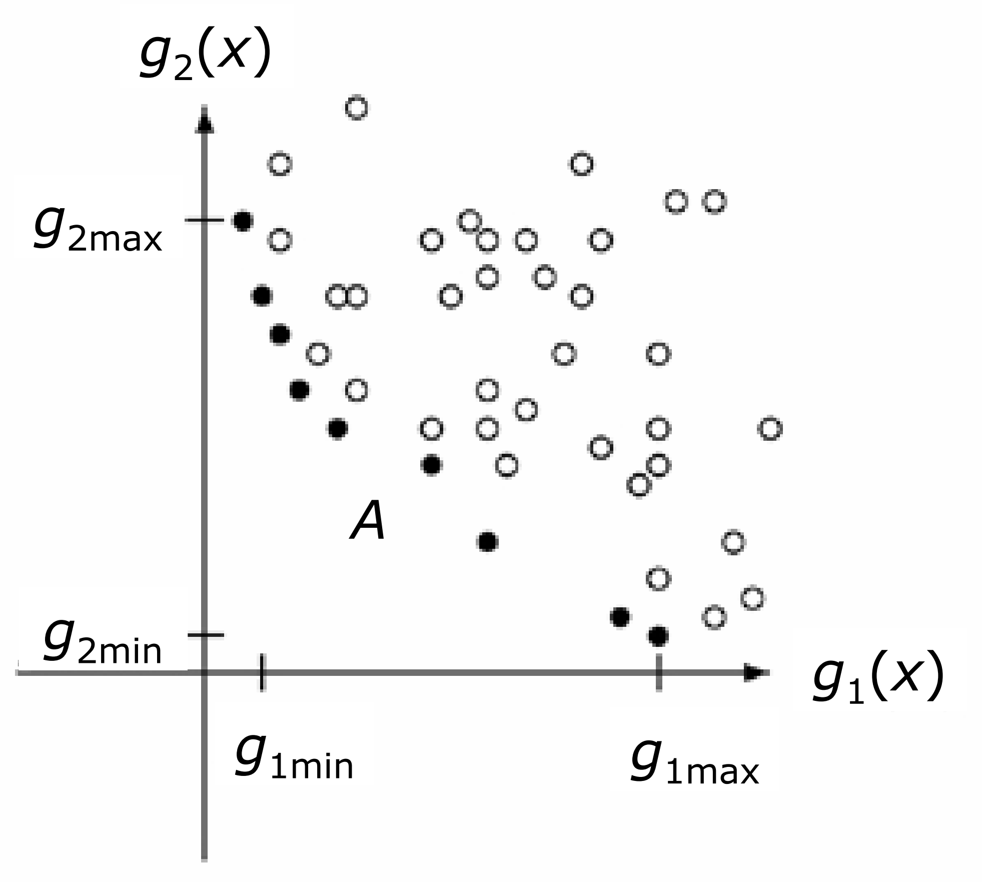
Multiple-criteria ranking using an additive utility function constructed via ordinal regresion : UTA method

Roman Słowiński

Poznań University of Technology, Poland

Problem statement

- Consider a finite set A of alternatives (actions, solutions) evaluated by n criteria from a consistent family $G=\{g_1,\ldots,g_n\}$



Problem statement

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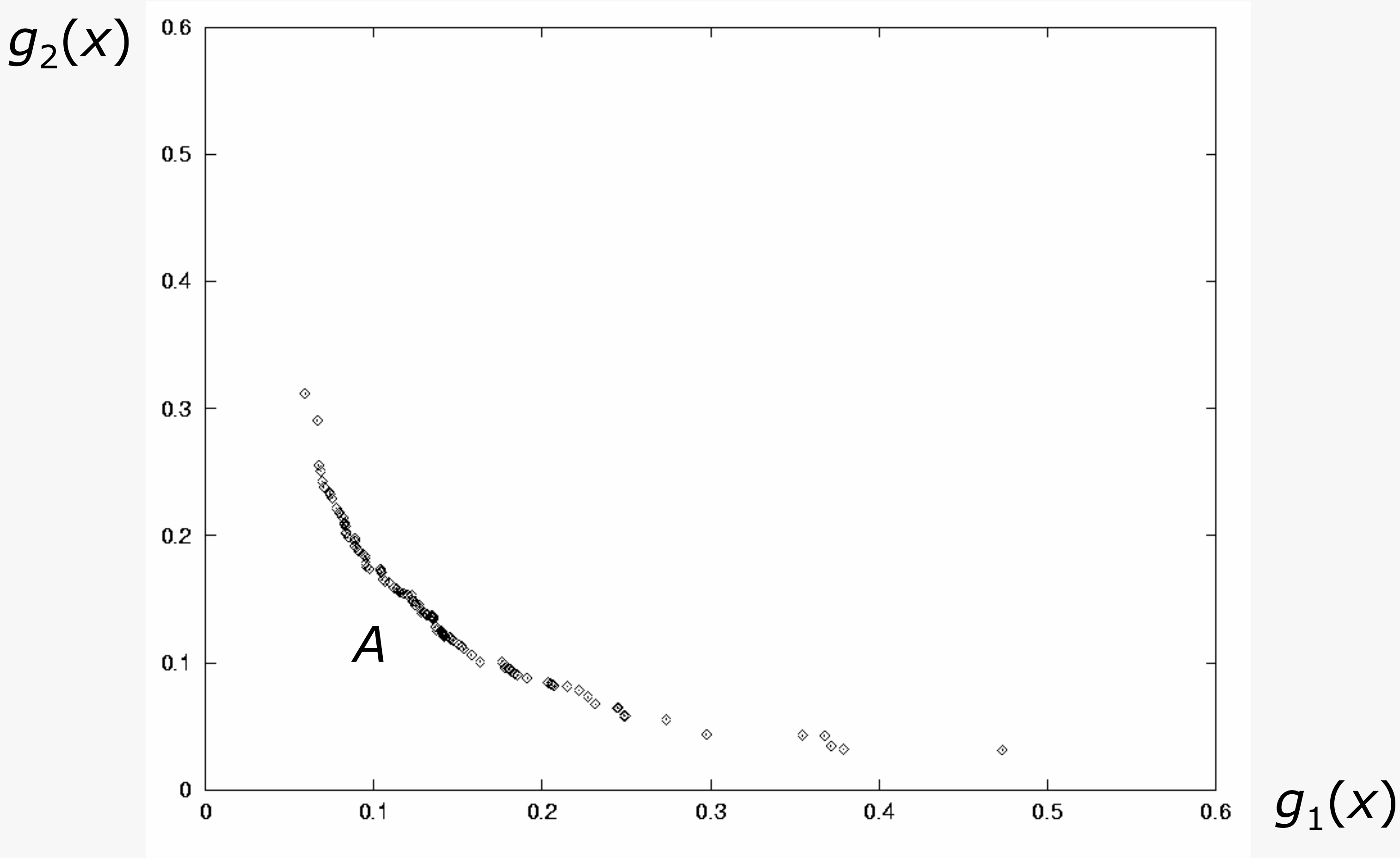
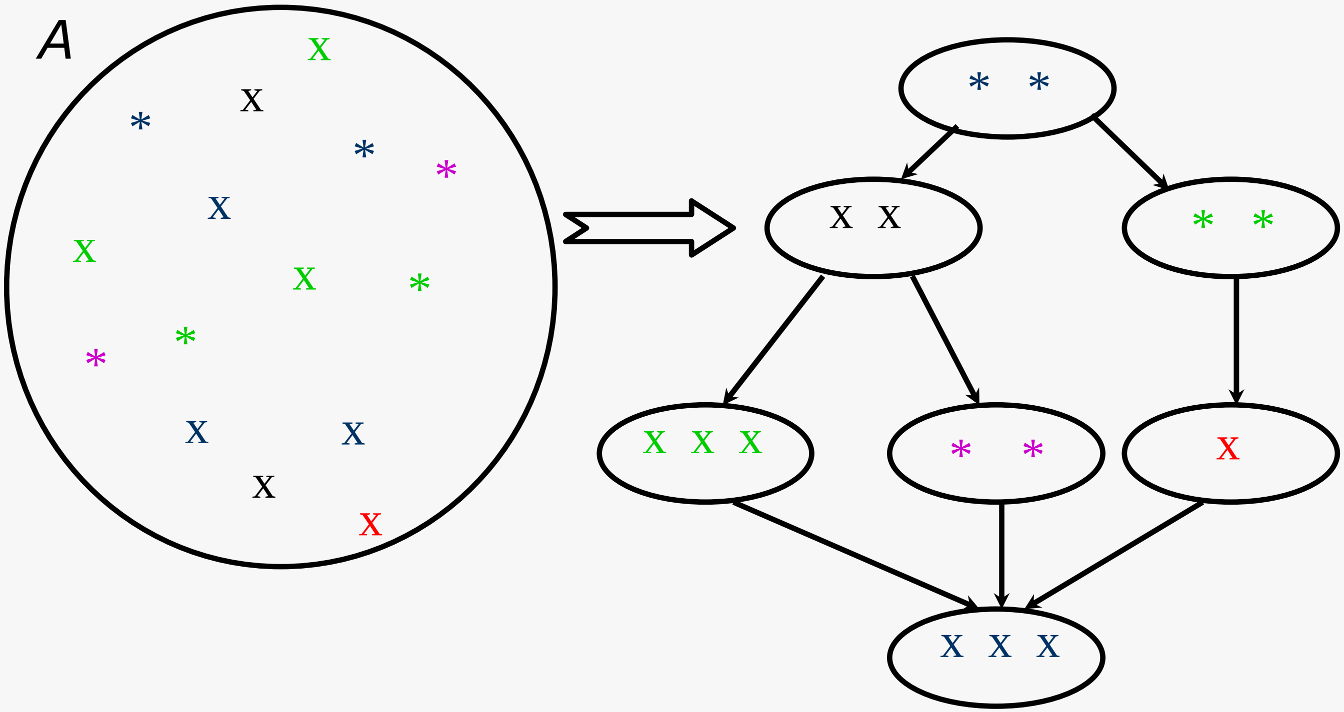


Problem statement

- Taking into account preferences of a Decision Maker (DM), **rank** all the alternatives of set *A* from the best to the worst



Basic concepts and notation

- X_i – domain of criterion g_i (X_i is finite or countably infinte)
- $X = \prod_{i=1}^{n} X_i$ – evaluation space
- $x,y \in X$ – profiles of alternatives in evaluation space
- $\succeq$ – weak preference (outranking) relation on X: for each $x,y \in X$

$x \succeq y \Leftrightarrow$ „x is at least as good as y"

$x \succ y \equiv [x \succeq y$ and *not* $y \succeq x] \Leftrightarrow$ „x is preferred to y"

$x \sim y \equiv [x \succeq y$ and $y \succeq x] \Leftrightarrow$ „x is indifferent to y"

Basic concepts and notation

- For simplicity: $X_i \subseteq \Re$, for all i=1,…,n
- For each g_i, $X_i=[\alpha_i, \beta_i]$ is the criterion evaluation scale, $\alpha_i \leq \beta_i$,
  where α_i and β_i, are the worst and the best (finite) evaluations, resp.
- Thus, A is a finite subset of X and

  $$g_i : A \rightarrow [\alpha_i, \beta_i] \subset \Re; \ a \rightarrow \boldsymbol{g}(a) \in \prod_{i=1}^{n} [\alpha_i, \beta_i]$$

  where $\boldsymbol{g}(a)$ is the vector of evaluations of alternative $a \in A$ on n criteria
- Additive value (or utility) function on X: for each $a \in X$

  $$U(a) = \sum_{i=1}^{n} u_i[g_i(a)]$$

  where u_i are non-decreasing marginal value functions, $u_i : X_i \rightarrow \Re$, i=1,...,n

Criteria aggregation model = preference model

- To solve a multicriteria decision problem one needs

  a criteria aggregation model, i.e. a preference model

- Traditional **aggregation** paradigm:

  The criteria aggregation model is first constructed and then applied on set *A* to get information about the comprehensive preference

- **Disaggregation-aggregation** (or regression) paradigm:

  The comprehensive preference on a subset $A^R \subseteq A$ is known a priori and a consistent criteria aggregation model is inferred from this information

Criteria aggregation model = preference model

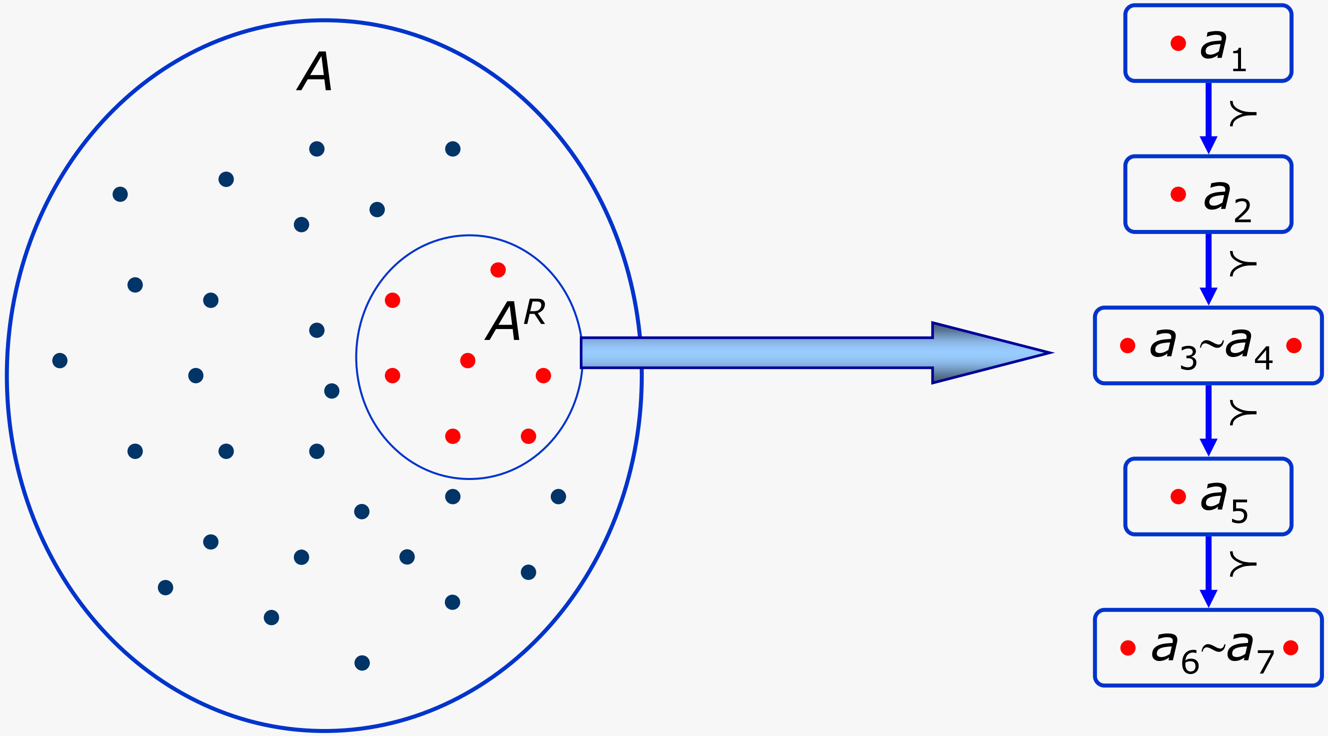
- The disaggregation-aggregation paradigam has been introduced to MCDA by Jacquet-Lagreze & Siskos (1982) in the UTA method – the inferred criteria aggregation model is the additive value function with piecewise-linear marginal value functions

- The disaggregation-aggregation paradigam is consistent with the „posterior rationality" principle by March (1988) and the inductive learning used in artificial intelligence and knowledge discovery

Principle of the UTA method (Jacquet-Lagreze & Siskos, 1982)

- The comprehensive preference information is given in form of a complete preorder on a subset of reference alternatives $A^R \subseteq A$,

  $A^R=\{a_1,a_2,\ldots,a_m\}$ – the reference alternatives are rearranged such that $a_k \succeq a_{k+1}$, $k=1,\ldots,m-1$



Principle of the UTA method

- Example:

  Let $A^R=\{a1, a2, a3\}$, $G=\{Gain\_1, Gain\_2\}$

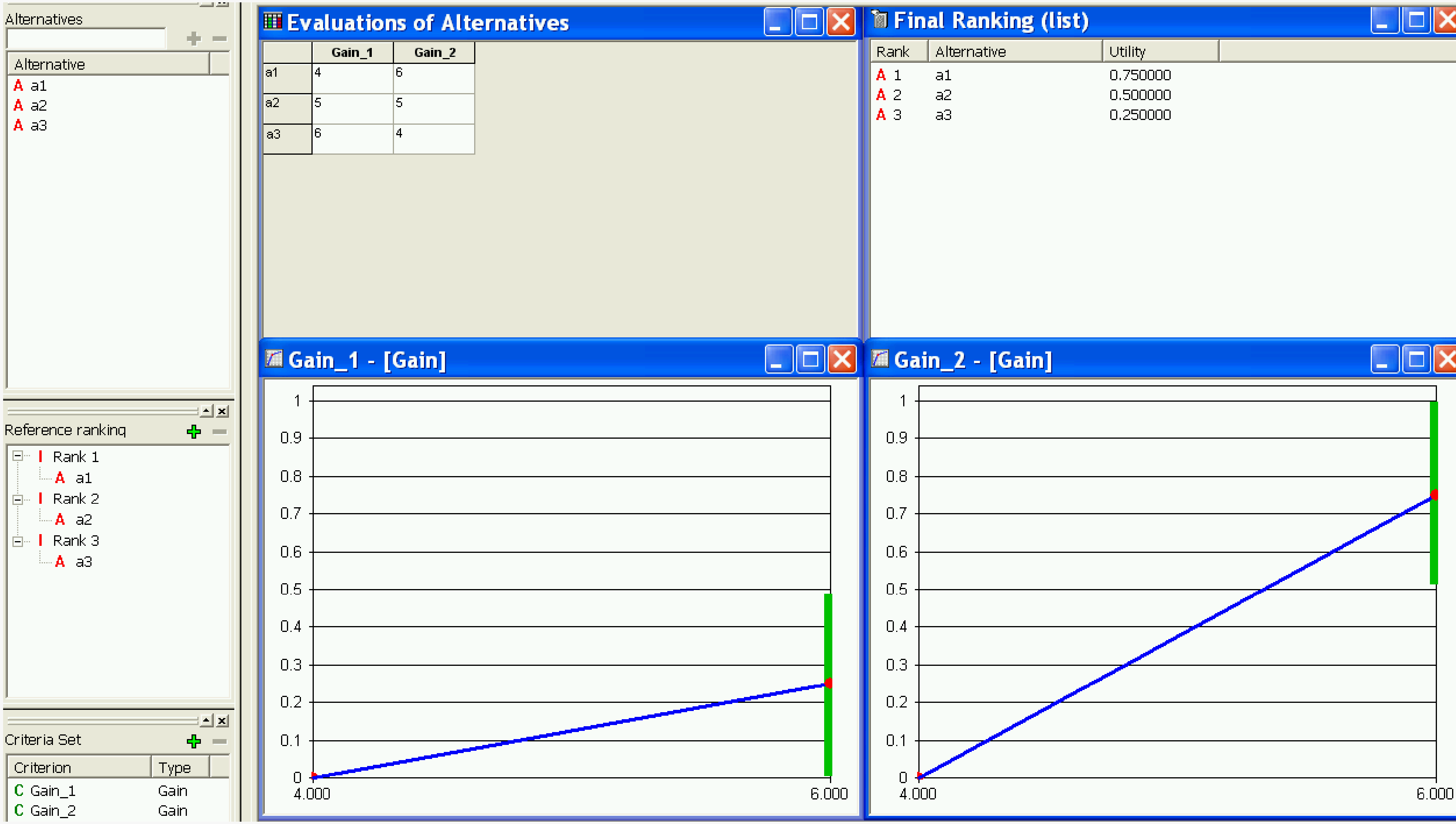
  Evaluation of reference alternatives on criteria *Gain_1*, *Gain_2*:

	Gain_1	Gain_2
*a*1	4	6
*a*2	5	5
*a*3	6	4

  Reference ranking:

$$\begin{array}{c} \boxed{\bullet\ a1} \\ \downarrow \succ \\ \boxed{\bullet\ a2} \\ \downarrow \succ \\ \boxed{\bullet\ a3} \end{array}$$

Principle of the UTA method



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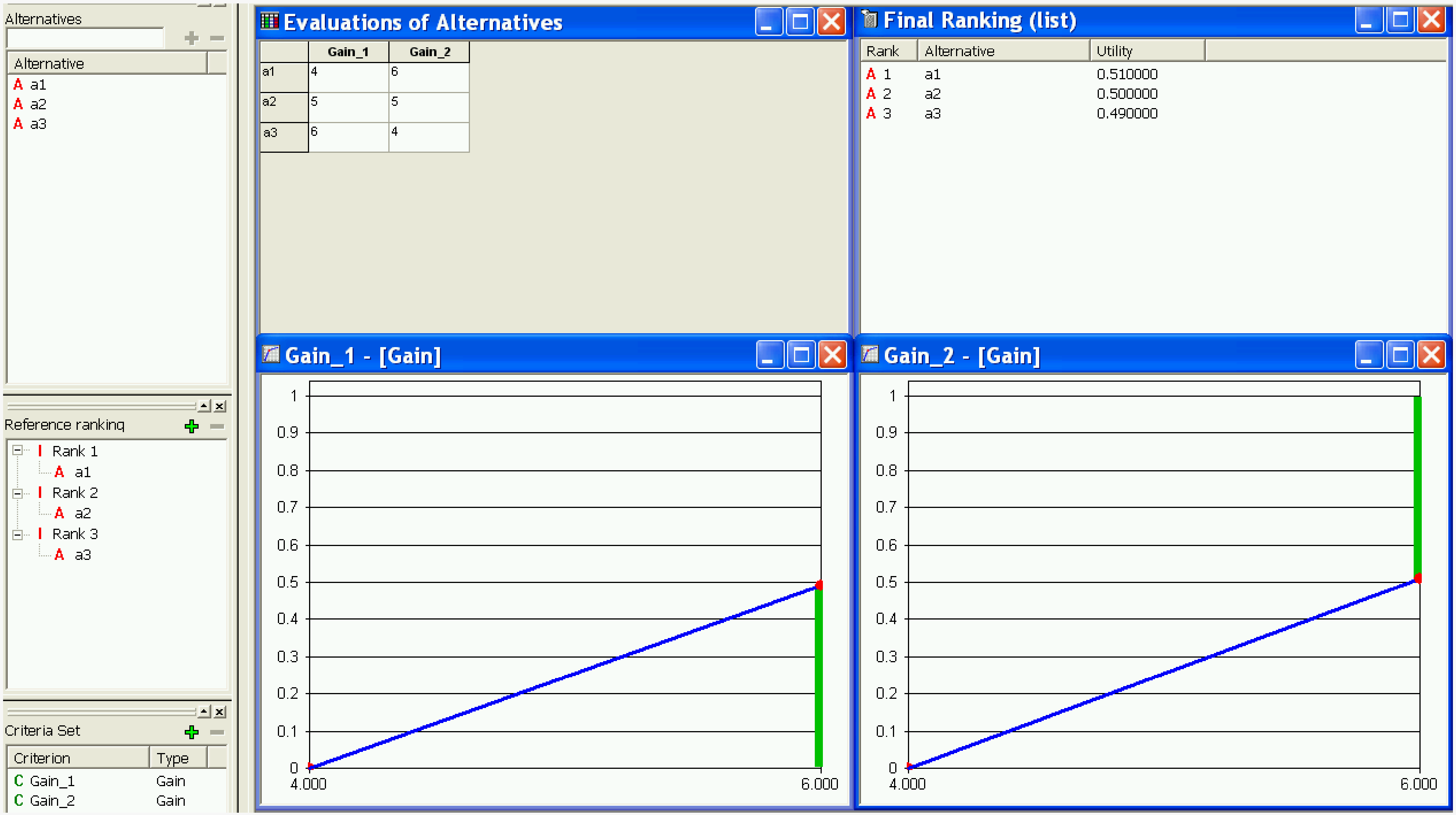
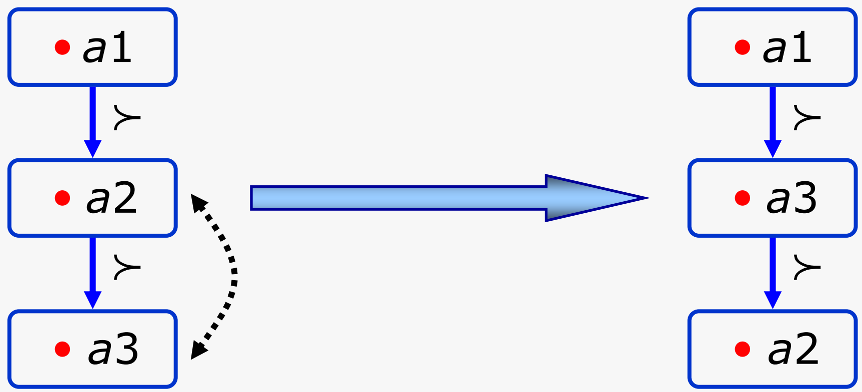


Principle of the UTA method

- Let's change the reference ranking:



- One linear piece per each marginal value function u_1, u_2 is not enough

	Gain_1	Gain_2
*a*1	4	6
*a*2	5	5
*a*3	6	4

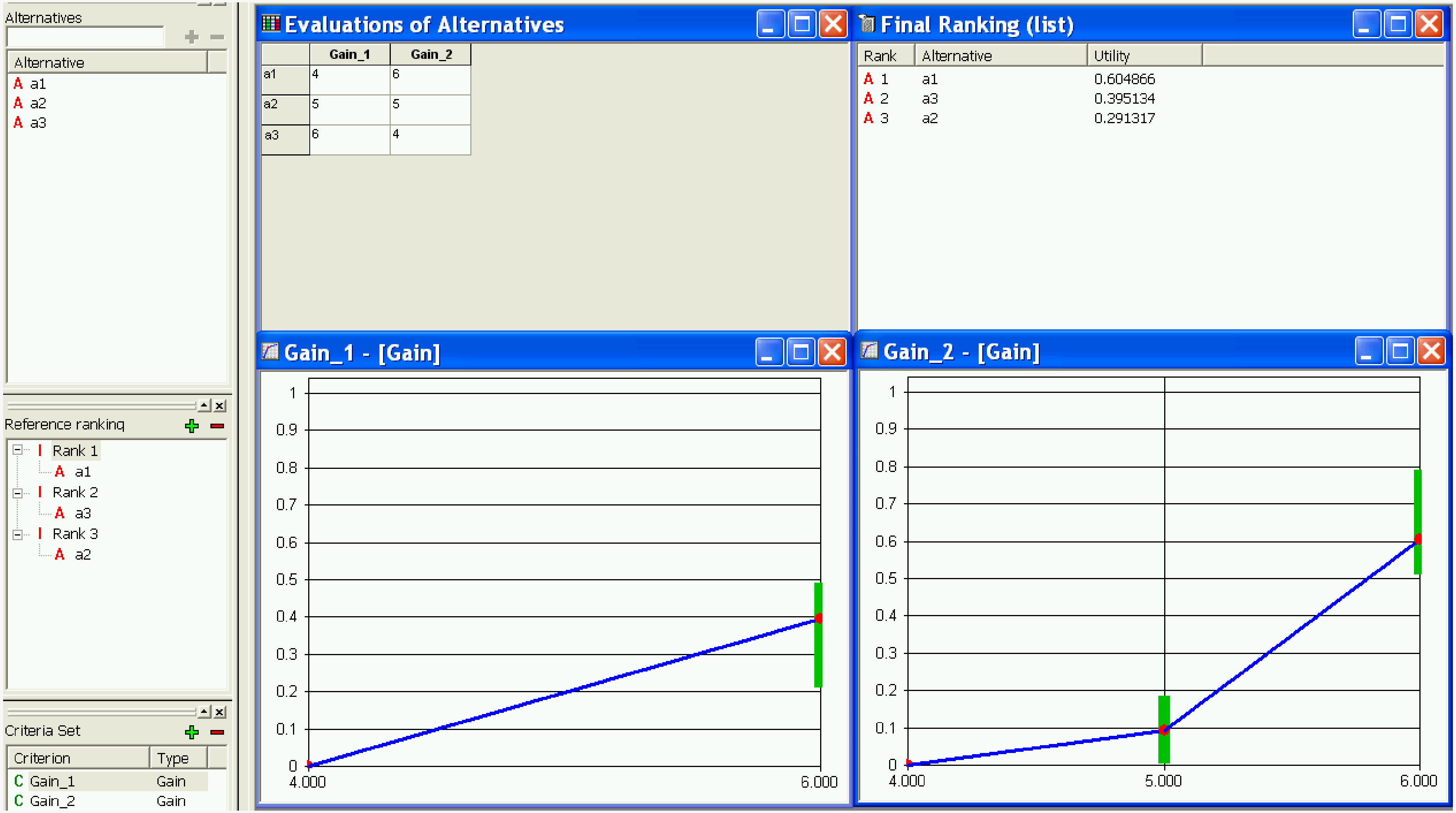
$u_1 = w_1 \times Gain\_1, \quad u_2 = w_2 \times Gain\_2, \quad U = u_1 + u_2$

For $a1 \succ a3$, $w_2 > w_1$,

but for $a3 \succ a2$, $w_1 > w_2$,

thus, marginal value functions cannot be linear

Principle of the UTA method



Principle of the UTA method

- The inferred value of each reference alternative $a \in A^R$:

$$U'[\mathbf{g}(a)] = U[\mathbf{g}(a)] - \sigma^+(a) + \sigma^-(a)$$

where

$U'[\mathbf{g}(a)] = \sum_{i=1}^{n} u'_i[g_i(a)]$ is a calculated value function,

$U[\mathbf{g}(a)] = \sum_{i=1}^{n} u_i[g_i(a)]$ is a value function compatible with the reference ranking,

σ^+ and σ^- are potential errors of over- and under-estimation of the compatible value function, respectively.

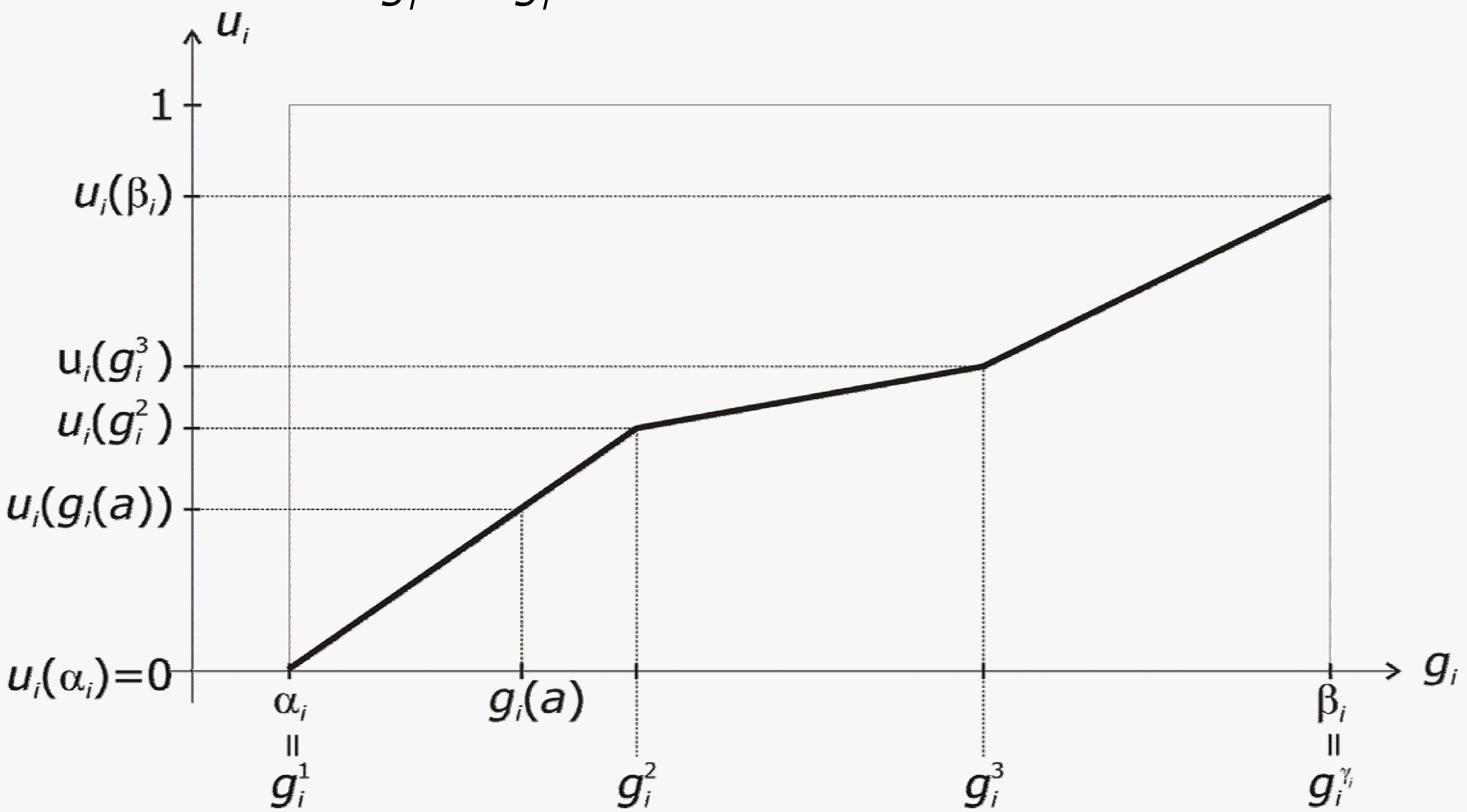
- The intervals $[\alpha_i, \beta_i]$ are divided into $(\gamma_i - 1)$ equal sub-intervals with the end points (i=1,...,n)

$$g_i^j = \alpha_i + \frac{j-1}{\gamma_i - 1}(\beta_i - \alpha_i), \quad j = 1,\ldots,\gamma_i$$

Principle of the UTA method

- The marginal value of alternative $a \in A$ is approximated by a linear interpolation: for $g_i(a) \in \left[g_i^j, g_i^{j+1}\right]$

$$u_i[g_i(a)] = u_i\left(g_i^j\right) + \frac{g_i(a) - g_i^j}{g_i^{j+1} - g_i^j}\left[u_i\left(g_i^{j+1}\right) - u_i\left(g_i^j\right)\right]$$



Principle of the UTA method

- Ordinal regression principle

  if $\Delta(a_k, a_{k+1}) = U'[\boldsymbol{g}(a_k)] - U'[\boldsymbol{g}(a_{k+1})]$ then one of the following holds

  $$\Delta(a_k, a_{k+1}) > 0 \text{ iff } a_k \succ a_{k+1}$$
  $$\Delta(a_k, a_{k+1}) = 0 \text{ iff } a_k \sim a_{k+1}$$

  N.B. In practice, „0” is replaced here by a small positive number that may influence the result

- Monotonicity of preferences

  $$u_i\left(g_i^{j+1}\right) - u_i\left(g_i^j\right) \geq 0 \quad j = 1, \ldots, \gamma_i - 1; \; i = 1, \ldots, n$$

- Normalization

  $$\sum_{i=1}^{n} u_i(\beta_i) = 1$$
  $$u_i(\alpha_i) = 0 \quad i = 1, \ldots, n$$

Principle of the UTA method

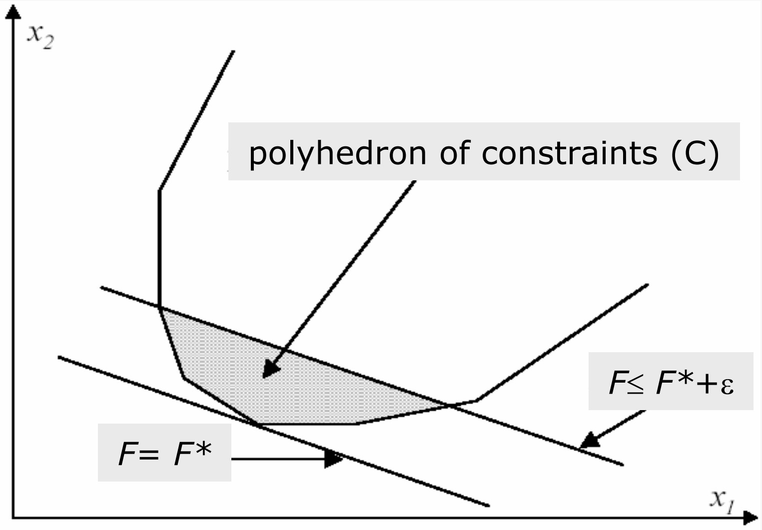
- The marginal value functions (breakpoint variables) are estimated by solving the LP problem

$$
\left.
\begin{array}{l}
\text{Min} \;\rightarrow\; F = \sum_{a \in A^R} \left[\sigma^+(a) + \sigma^-(a)\right] \\
\text{subject to} \\
\Delta(a_k, a_{k+1}) > 0 \;\text{ iff }\; a_k \succ a_{k+1} \\
\Delta(a_k, a_{k+1}) = 0 \;\text{ iff }\; a_k \sim a_{k+1} \\
u_i\left(g_i^{j+1}\right) - u_i\left(g_i^j\right) \geq 0 \quad j = 1, \ldots, \gamma_i - 1; \;\; i = 1, \ldots, n \\
\sum_{i=1}^{n} u_i(\beta_i) = 1 \\
u_i(\alpha_i) = 0 \quad i = 1, \ldots, n \\
u_i\left(g_i^j\right) \geq 0, \;\; \sigma^+(a) \geq 0, \;\; \sigma^-(a) \geq 0, \;\; \forall a \in A^R, \;\; \forall i \;\text{ and }\; j
\end{array}
\right\} \text{(C)}
$$

Principle of the UTA method

- If $F^*=0$, then the polyhedron of feasible solutions for $u_i(g_i)$ is not empty and there exists at least one value function $U[\mathbf{g}(a)]$ compatible with the complete preorder on A^R
- If $F^*>0$, then there is no value function $U[\mathbf{g}(a)]$ compatible with the complete preorder on A^R – *three possible moves:*
  - increasing the number of linear pieces γ_i for $u_i(g_i)$
  - revision of the complete preorder on A^R
  - post optimal search for the best function with respect to Kendall's τ in the area $F \leq F^* + \varepsilon$

Jacquet-Lagreze & Siskos (1982)



Współczynnik Kendalla

- Do wyznaczania odległości między preporządkami stosuje się miarę Kendalla
- Przyjmijmy, że mamy dwie macierze kwadratowe R i R^* o rozmiarze $m \times m$, gdzie $m = |A^R|$, czyli m jest liczbą wariantów referencyjnych
  - macierz R jest związana z porządkiem referencyjnym podanym przez decydenta,
  - macierz R^* jest związana z porządkiem dokonanym przez funkcję użyteczności wyznaczoną z zadania PL (zadania regresji porządkowej)
- Każdy element macierzy R, czyli r_{ij} (i, j=1,..,m), może przyjmować wartości:

$$r_{ij} = \begin{cases} 0, & \text{gdy } i = j \text{ } lub \text{ } a_j \succ a_i \\ 0.5, & \text{gdy } a_j \sim a_i \\ 1, & \text{gdy } a_i \succ a_j \end{cases}$$

- To samo dotyczy elementów macierzy R^*
- Tak więc w każdej z tych macierzy kodujemy pozycję (w porządku) wariantu a względem wariantu b

Współczynnik Kendalla

- Następnie oblicza się *współczynnik Kendalla* τ:

$$\tau = 1 - 4 \cdot \frac{d_k(R, R^*)}{m(m-1)}$$

gdzie $d_k(R,R^*)$ jest *odległością Kendalla* między macierzami R i R^*:

$$d_k(R, R^*) = \frac{1}{2} \cdot \sum_{i=1}^{m} \sum_{j=1}^{m} \left| r_{ij} - r_{ij}^* \right|$$

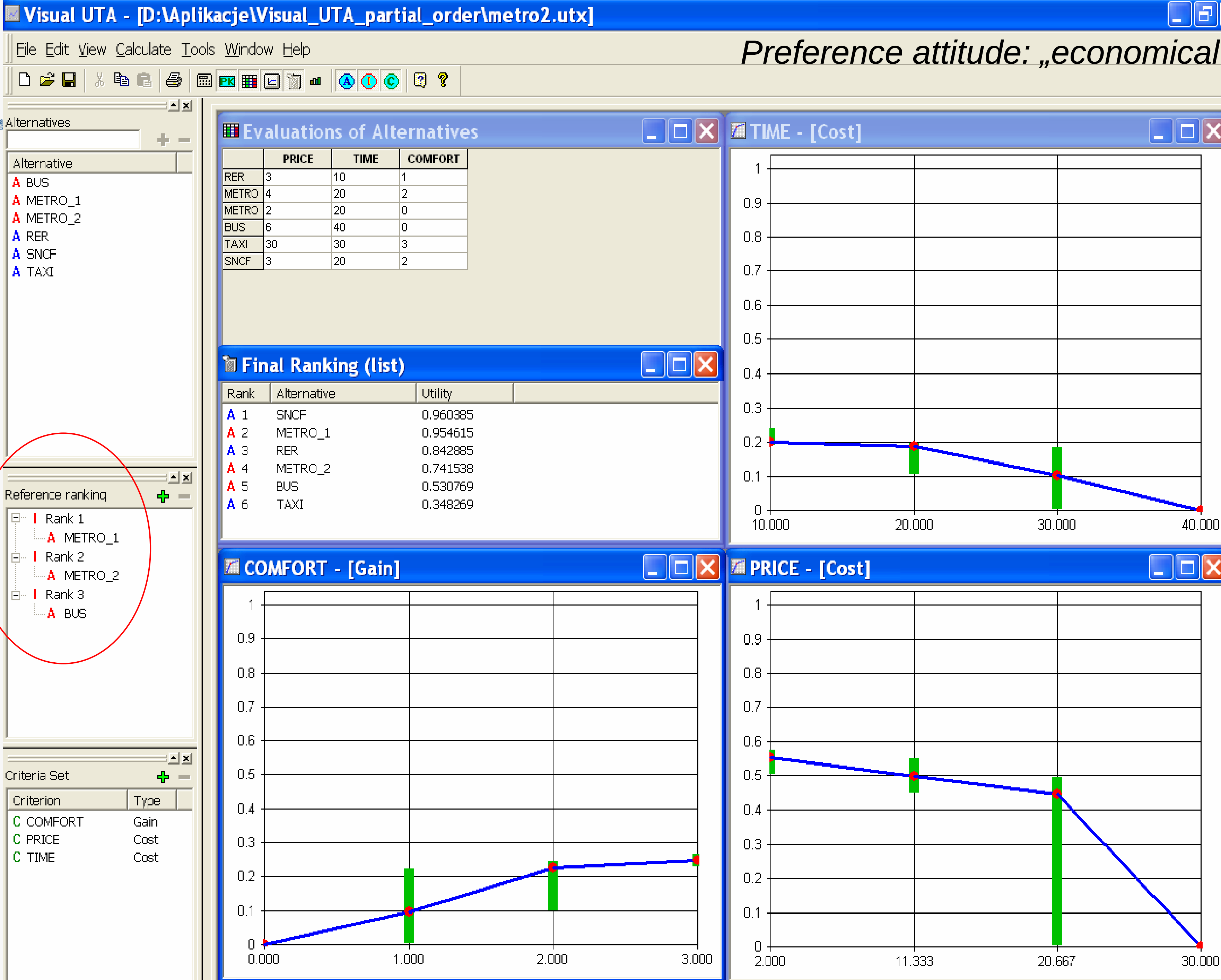
- Stąd $\tau \in \langle -1, 1 \rangle$
- Jeżeli $\tau = -1$, to oznacza to, że porządki zakodowane w macierzach R i R^* są zupełnie odwrotne, np. macierz R koduje porządek $a \succ b \succ c \succ d$, a macierz R^* porządek $d \succ c \succ b \succ a$
- Jeżeli $\tau = 1$, to zachodzi całkowita zgodność porządków z obydwu macierzy. W tej sytuacji błąd estymacji funkcji użyteczności $F^*=0$
- W praktyce funkcję użyteczności akceptuje się, gdy $\tau \geq 0.75$

Example of UTA$^+$

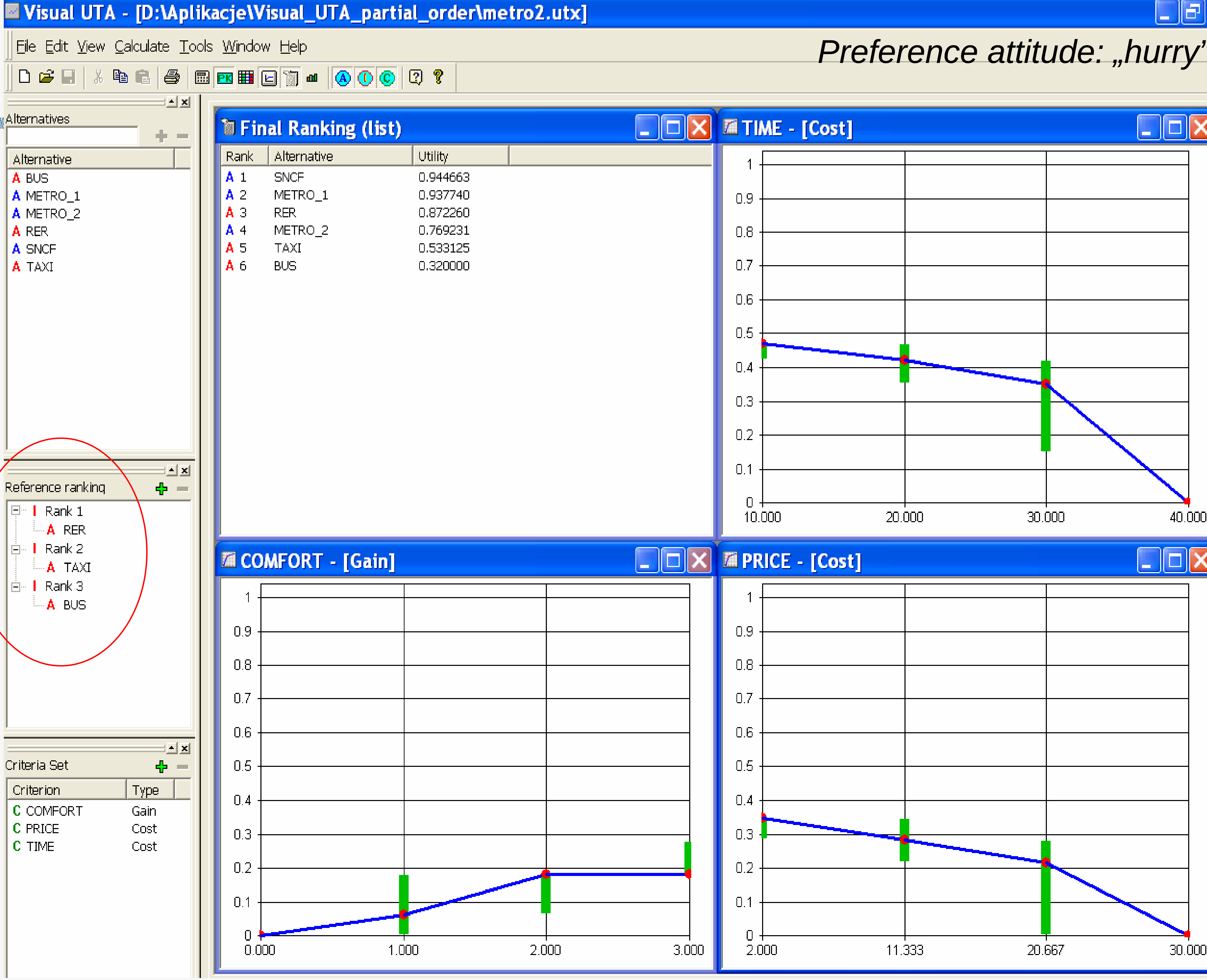
- Ranking of 6 means of transportation

	PRICE	TIME	COMFORT
RER	3	10	1
METRO1	4	20	2
METRO2	2	20	0
BUS	6	40	0
TAXI	30	30	3
SNCF	3	20	2

Preference attitude: „economical”



Preference attitude: „hurry”



Preference attitude: „hurry”