Preference Modelling and Decision Support

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Decision problem

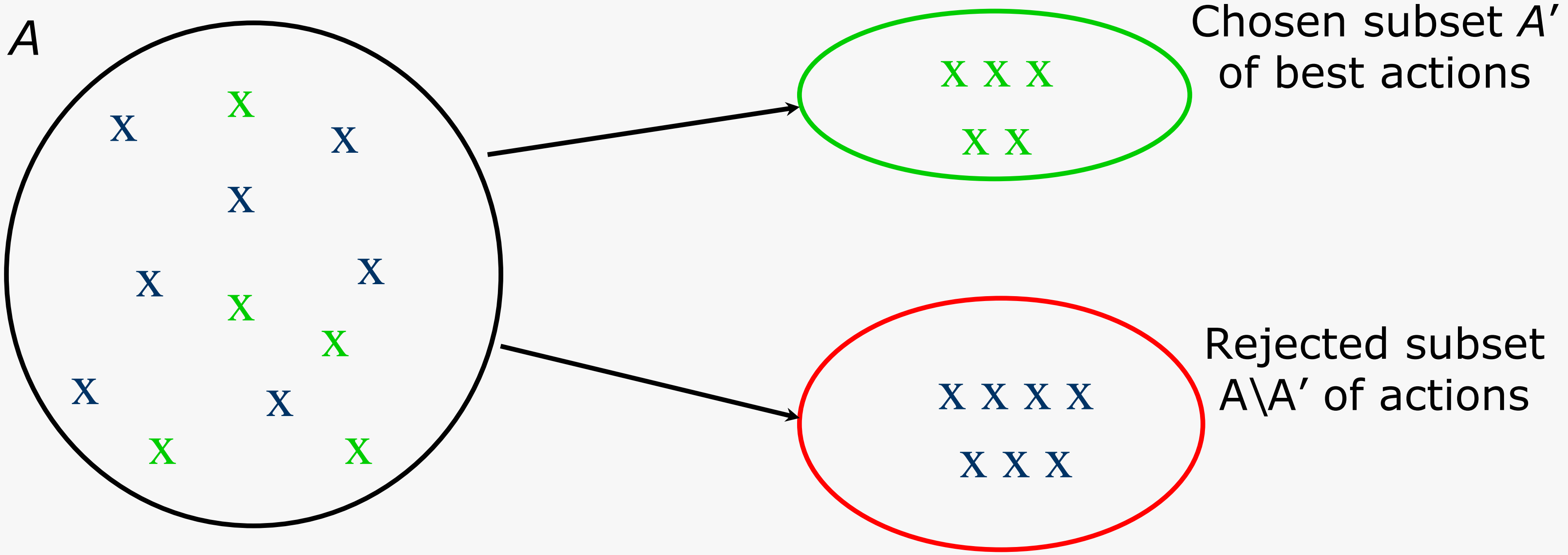
- There is a goal or goals to be attained
- There are many alternative ways for attaining the goal(s) – they consititute a set of actions *A* (alternatives, solutions, variants, …)
- A decision maker (DM) may have one of following questions with respect to set *A*:

  P_α: *How to choose the best action* ?

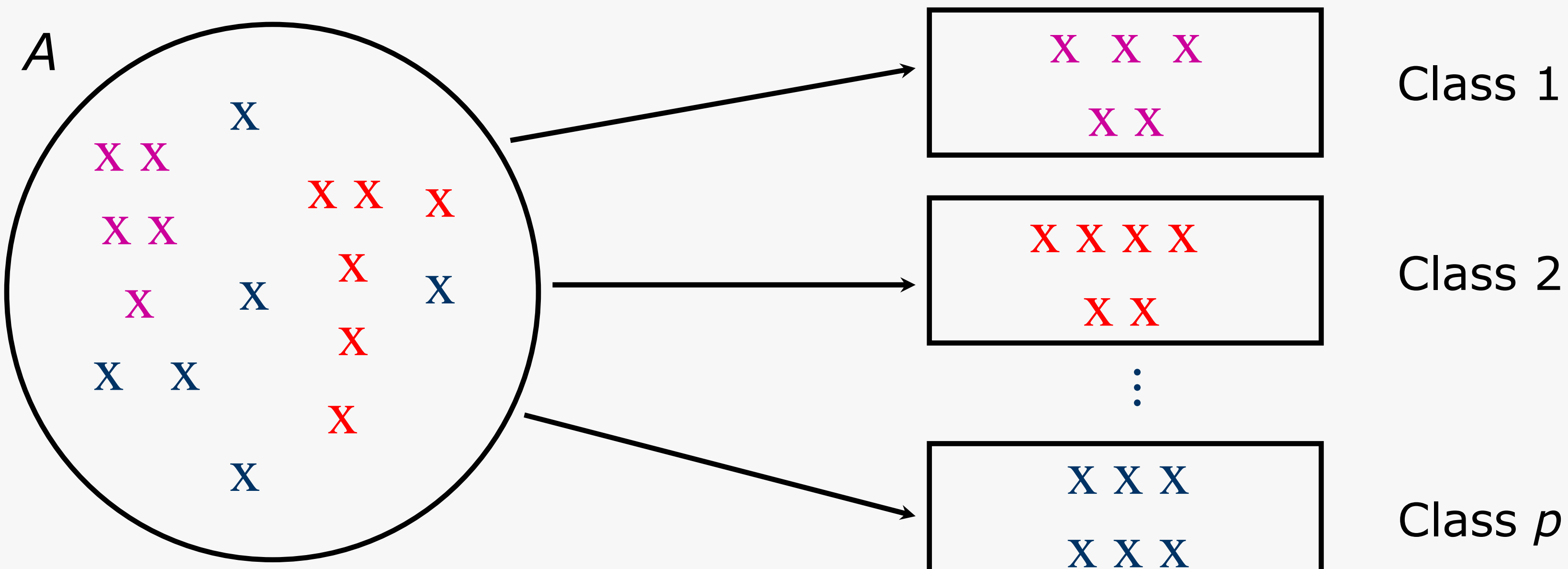
  P_β : *How to classify actions into pre-defined decision classes* ?

  P_γ : *How to order actions from the best to the worst* ?

P_α : Choice problem (optimization)



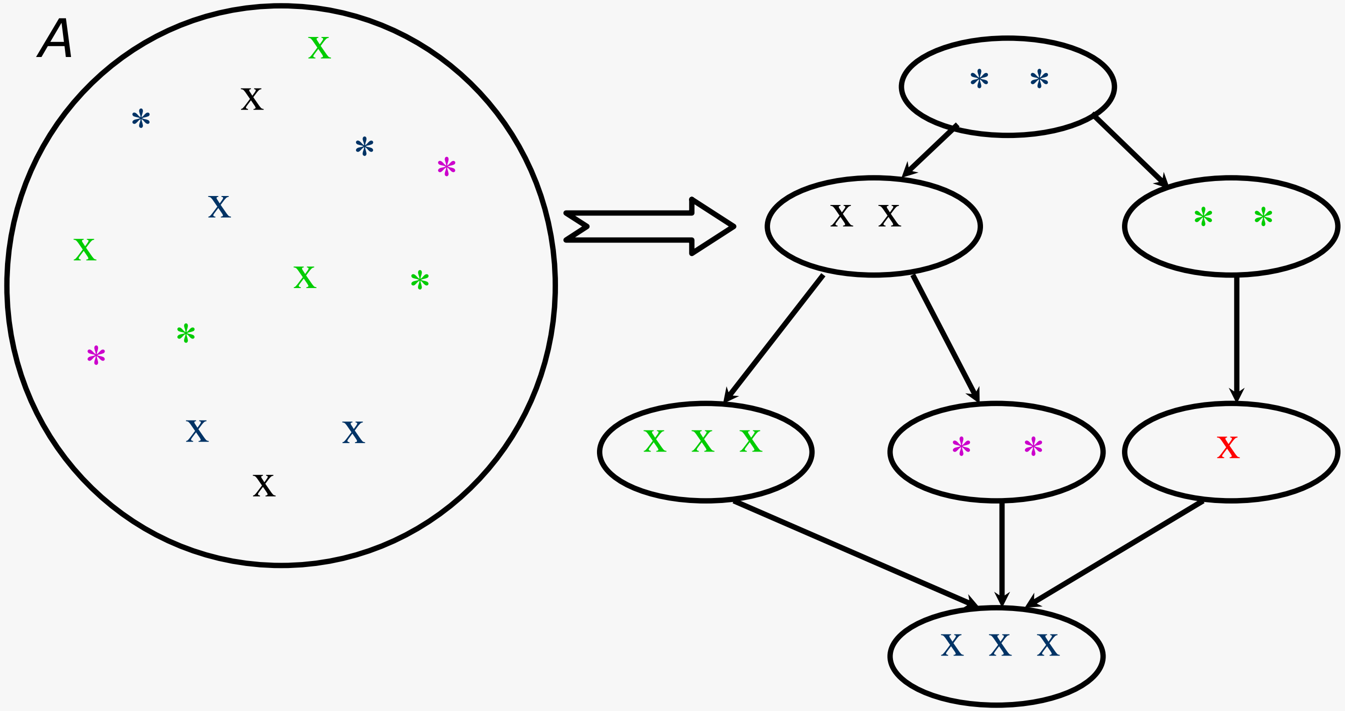
P_β : Classification problem (sorting)



Class 1 $\succ$ Class 2 $\succ$ … $\succ$ Class p

P_γ : Ordering problem (ranking)

Partial or complete ranking
of actions



Coping with multiple dimensions in decision support

- Questions P_α, P_β , P_γ are followed by new questions:

DM: who is the decision maker and how many they are ?

MC: what are the evaluation criteria and how many they are ?

RU: what are the consequences of actions and are they deterministic or uncertain (single state of nature with P=1 or multiple states of nature with different $P \leq 1$) ?

	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ
DM	1	m	1	1	m	m	1	m
MC	1	1	n	1	n	1	n	n
RU	1	1	1	RU	1	RU	RU	RU

Optimization
Sorting
Ranking

Theory of
Social
Choice (TSC)

Multi-Criteria
Decision
Making (MCDM)

Decision under
Risk and
Uncertainty (DRU)

Solution „philosophically" simple

For a conflict between dimensions DM, MC, RU, the decision problem has no solution at this stage (*ill-posed problem*)

Translation table

	Theory of Social Choice	Multi-Criteria Decision Making	Decision under Risk and Uncertainty
Element of set *A*	Candidate	Action	Act
Dimension of evaluation space	Voter	Criterion	Probability of an outcome
Objective information about elements of *A*	Dominance relation	Dominance relation	Stochastic dominance relation

TSC

	Voters	
Cand.	V_1	V_2
a	3	1
b	1	2
c	2	3

MCDA

	Criteria	
Action	Time	Cost
a	3	1
b	1	2
c	2	3

DRU

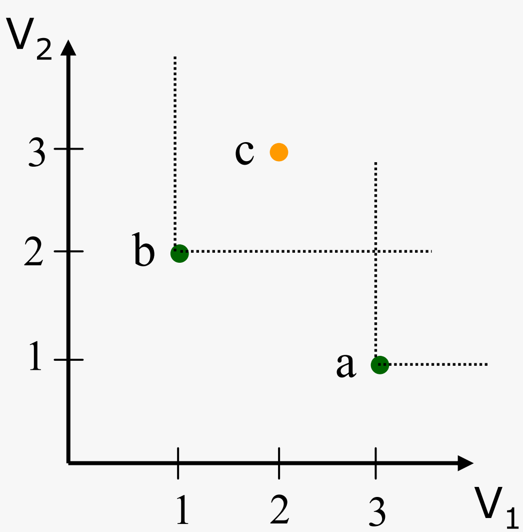
	Probability of gain	
Act	Gain$\geq G_1$	Gain$\geq G_2$
a	0.7	0.6
b	1.0	0.5
c	0.8	0.4

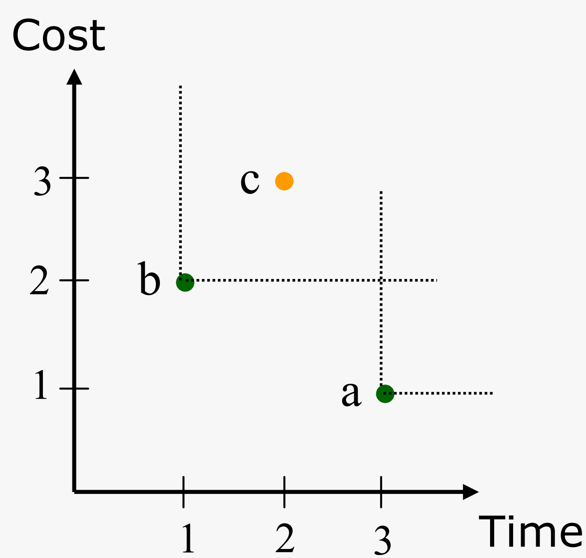
$V_1 : b \succ c \succ a$

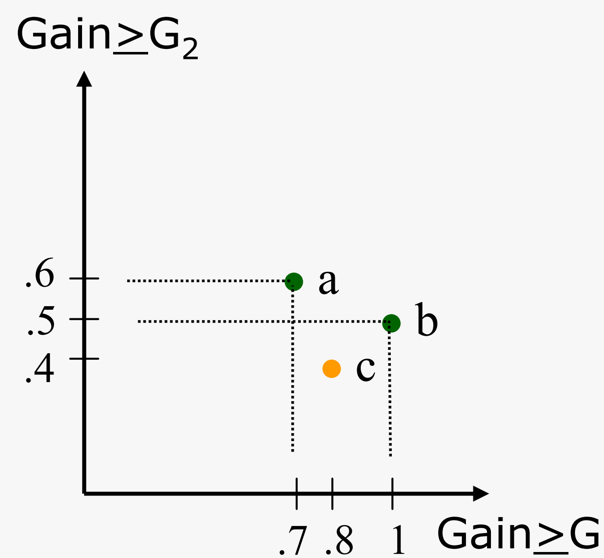
$V_2 : a \succ b \succ c$

- non-dominated
- dominated

$G_1 < G_2$







Preference modelling

- Dominance relation is too poor – it leaves many actions non-comparable
- One can „enrich" the dominance relation, using preference information elicited from the Decision Maker
- Preference information permits to built a preference model that aggregates the vector evaluations of elements of *A*
- Due to the aggregation, the elements of *A* become more comparable
- A proper exploitation of the preference relation in *A* leads to a final recommendation in terms of the best choice, classification or ranking
- We will concentrate on **Multi-Criteria Decision Making**, i.e. dimension = criterion

Preference modeling

- Three families of **preference models**:
  - Function, e.g. additive utility function (Debreu 1960, Luce & Tukey 1964)

$$U(a) = \sum_{i=1}^{n} u_i[g_i(a)]$$

  - Relational system, e.g. outranking relation S or fuzzy relation (Roy 1968)

aSb = “a is at least as good as b”

  - Set of decision rules,

e.g. “If $g_i(a) \geq r_i$ & $g_j(a) \geq r_j$ & … $g_h(a) \geq r_h$, then $a \rightarrow$ *Class t* or higher”

“If $\Delta_i(a,b) \geq s_i$ & $\Delta_j(a,b) \geq s_j$ & … $\Delta_h(a,b) \geq s_h$, then aSb”

- The rule model is the most general of all three

Greco, S., Matarazzo, B., Słowiński, R.: Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules. *European J. of Operational Research*, 158 (2004) no. 2, 271-292

What is a criterion ?

- Criterion is a real-valued function g_i defined on A, reflecting a worth of actions from a particular point of view, such that in order to compare any two actions $a,b \in A$ from this point of view it is sufficient to compare two values: $g_i(a)$ and $g_i(b)$

- Scales of criteria:

  - Ordinal scale – only the order of values matters; a distance in ordinal scale has no meaning of intensity, so one cannot compare differences of evaluations (e.g. school marks, customer satisfaction, earthquake scales)

  - Cardinal scales – a distance in ordinal scale has a meaning of intensity:

    - Interval scale – „zero" in this scale has no absolute meaning, but one can compare differences of evaluations (e.g. Celsius scale)

    - Ratio scale – „zero" in this scale has an absolute meaning, so a ratio of evaluations has a meaning (e.g. weight, Kelvin scale)

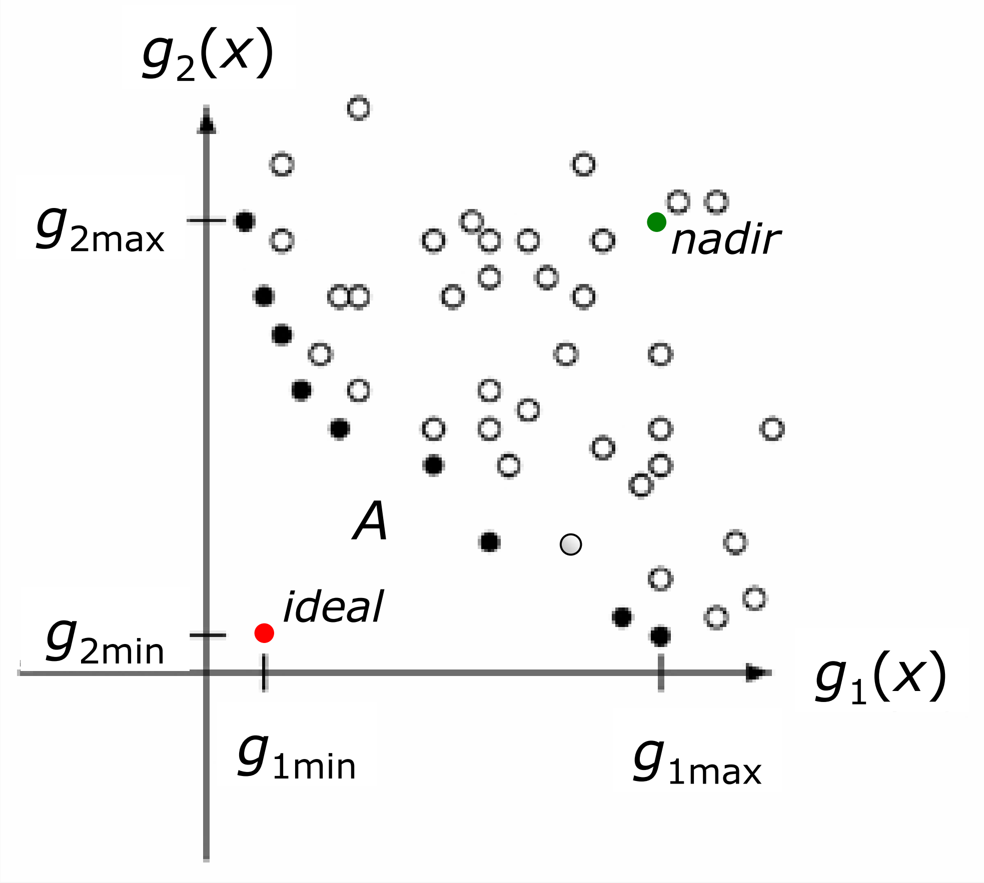
What is a consistent family of criteria ?

- A family of criteria $G=\{g_1,...,g_n\}$ is consistent if it is:
  - Complete – if two actions have the same evaluations on all criteria, then they have to be indifferent, i.e.

    if for any $a,b\in A$, there is $g_i(a)\sim g_i(b)$, $i=1,\ldots,n$, then $a\sim b$
  - Monotonic – if action a is preferred to action b ($a\succ b$), and there is action c, such that $g_i(c)\succeq g_i(a)$, $i=1,\ldots,n$, then $c\succ b$
  - Non-redundant – elimination of any criterion from the family G should violate at least one of the above properties

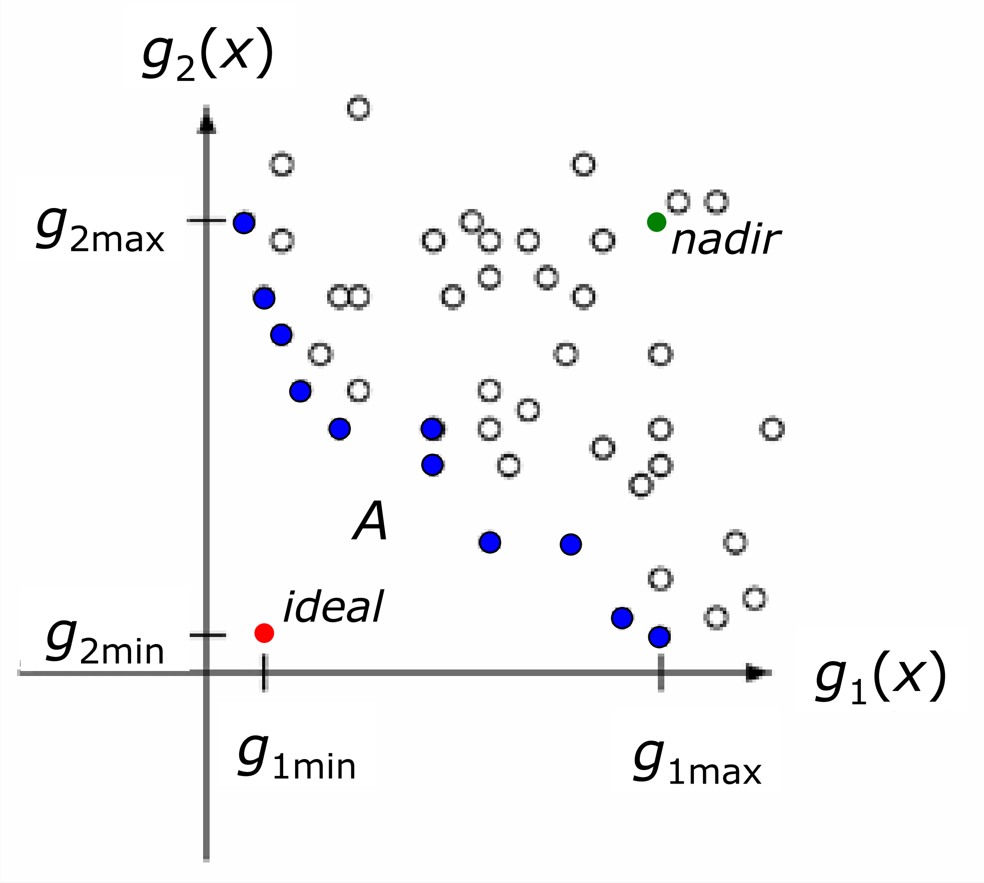
Dominance relation

- Action $a \in A$ is non-dominated (Pareto-optimal) if and only if there is no other action $b \in A$ such that $g_i(b) \succeq g_i(a)$, $i=1,\ldots,n$, and on at least one criterion $j=\{1,\ldots,n\}$, $g_j(b) \succ g_j(a)$



Dominance relation

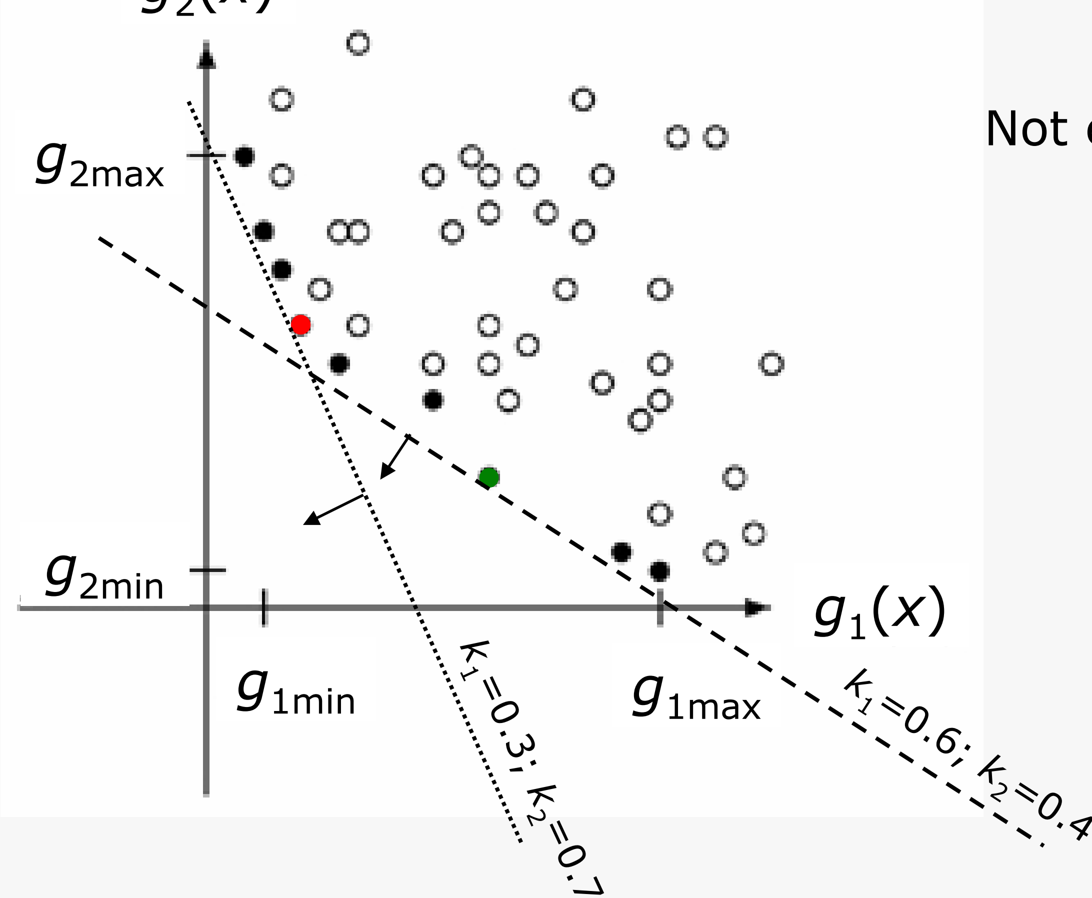
- Action $a \in A$ is weakly non-dominated (weakly Pareto-optimal) if and only if there is no other action $b \in A$ such that $g_i(b) \succ g_i(a)$, $i=1,\ldots,n$,



Preference modeling using a utility function U

- The most intuitive model: $U(a) = \sum_{i=1}^{n} k_i g_i(a)$

The preference information = *trade-off weights* k_i



Not easy to elicit and, moreover, criteria must be independent

Easy exploitation of the preference relation induced by U in A

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} k_i g_i(a) \geq \sum_{i=1}^{n} k_i g_i(b)$$

Preference modelling using a „weighted sum"

Example: let the weights be k_1=0.6, k_2 =0.4

- The weighted sum allows trade-off (compensation) between criteria:

  $U(g_1, g_2) = U(g_1+1, g_2-\boldsymbol{x})$, i.e.

  $g_1 \times k_1 + g_2 \times k_2 = (g_1+1) \times k_1 + (g_2-\boldsymbol{x}) \times k_2$ or $k_1 = \boldsymbol{x} \times k_2$, thus

  - $\boldsymbol{x} = k_1/k_2$ – change on criterion g_2, able to compensate a change by 1 on criterion g_1, i.e., $\boldsymbol{x}=1.5$

- Analogously, $\boldsymbol{x'} = k_2/k_1$ – change on criterion g_1, able to compensate a change by 1 on criterion g_2, i.e., $\boldsymbol{x'}=0.67$

- For a scale of criteria from 0 to h, it makes sense that:

$$0 \leq k_1/k_2 \leq h \quad \text{and} \quad 0 \leq k_2/k_1 \leq h$$

Other properties of a „weighted sum"

- The weights and thus the trade-offs are constant for the whole range of variation of criteria values
- The „weighted sum" and, more generally, an additive utility function requires that criteria are independent in the sense of preferences, i.e. $u_i(a)=g_i \times k_i$ does not change with a change of $g_j(a)$, $j=1,\ldots,n$; $j \neq i$
- In other words, this model cannot represent the following preferences:

Car	(↓) Gas consumption	(↓) Price	(↑) Comfort
a	5	90	5
b	9	90	9
c	5	50	5
d	9	50	9

$b \succ a$ while $c \succ d$

It requires that:

if $b \succ a$ then $d \succ c$

Preference modeling using more genral utility function *U*

- Additive difference model (Tversky 1969, Fishburn 1991)

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} \varphi_i \{u_i[g_i(a)] - u_i[g_i(b)]\} \geq 0$$

- Transitive decomposable model (Krantz et al. 1971)

$$a \succeq b \Leftrightarrow f\{u_1[g_1(a)],\ldots,u_n[g_n(a)]\} \geq f\{u_1[g_1(b)],\ldots,u_n[g_n(b)]\}$$

f: $\boldsymbol{R}^n \to \boldsymbol{R}$, non-decreasing in each argument

- Non-transitive additive model (Bouyssou 1986, Fishburn 1990, Vind 1991)

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} v_i[g_i(a), g_i(b)] \geq 0$$

v_i: $\boldsymbol{R}^2 \to \boldsymbol{R}$, $i=1,\ldots,n$, non-decreasing in the first and non-increasing in the second argument

- Non-transitive non-additive model (Fishburn 1992, Bouyssou & Pirlot 1997)

$$a \succeq b \Leftrightarrow f\{v_1[g_1(a), g_1(b)],\ldots,v_n[g_n(a), g_n(b)]\} \geq 0$$