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QUOTATONE APPORTIONMENT METHODS*§

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The problem of apportionment is that of allocating an integer number of seats "proportionally" among a set of states or regions as a fraction of their populations. An apportionment method satisfies quota if it accords to each state the exactly proportional (rational) number of seats due it rounded up or rounded down. A method is house monotone if no state's allocation goes down when the total number of seats to be distributed goes up.

This paper gives a simple characterization of all house monotone methods satisfying quota. Further, a manner of exposition is formulated which unites several key house monotone apportionment methods, thus showing clearly their connections.

1. Introduction. Let $\mathbf{p} = (p_1, \ldots, p_s)$ be the populations of s states, where each $p_i > 0$, and h $\geqslant 0$ the number of seats in the house. The problem is to find, for any $\mathbf{p}$ and all h, an *apportionment for* h, that is, an s-tuple of nonnegative integers $\mathbf{a} = (a_1, \ldots, a_s)$ whose sum is h. A *solution* of the apportionment problem is therefore a function $\mathbf{f}$ which to every $\mathbf{p}$ and all h associates a unique apportionment for h, $a_i = f_i(\mathbf{p}, h) \geqslant 0$ where $\sum_i a_i = h$. If $\mathbf{f}$ is a solution and h a house size then $\mathbf{f}^h$ is the function $\mathbf{f}$ restricted to the domain $(\mathbf{p}, h')$, where $0 \leqslant h' \leqslant h$. $\mathbf{f}^h$ is called a *solution up to* h and $\mathbf{f}$ is called an *extension* of $\mathbf{f}^h$. A specific apportionment "method" may give several different solutions, for "ties" may occur when using it, for example when two states have identical populations. For this reason it is useful to define an *apportionment method* **M** as a nonempty set of solutions. A method **M** is the unique one satisfying given properties if any other collection of solutions with these properties is a collection of **M**-solutions.

Let $p = \sum_i p_i$ be the total population. The *exact quota* of state j is $q_j(\mathbf{p}, h) = p_j h/p$, its *lower quota* is $\lfloor q_j(p, h) \rfloor$ (the largest integer less than or equal to q_j), and *upper quota* is $\lceil q_j(\mathbf{p}, h) \rceil$ (the least integer greater than or equal to q_j). An apportionment method is said to *satisfy lower quota* if, for each of its solutions $\mathbf{f}$, $f_i(\mathbf{p}, h) \geqslant \lfloor q_i(\mathbf{p}, h) \rfloor$, to *satisfy upper quota* if $f_i(\mathbf{p}, h) \leqslant \lceil q_i(\mathbf{p}, h) \rceil$, and to *satisfy quota* if it satisfies both lower and upper quota. A method is said to be *house monotone* if, for each of its solutions $\mathbf{f}$, $\mathbf{f}(\mathbf{p}, h + 1) \geqslant \mathbf{f}(\mathbf{p}, h)$. A method **M** is *quotatone* if **M** is house monotone and satisfies quota.

The existence of a quotatone method was first established in [1], [4]. This method, called the Quota Method, was also shown to be the *unique* such method satisfying a certain property of *consistency* (subject to satisfying quota)—unique meaning that any method with the stated properties must be a set of Quota solutions.

On the other hand, not every quota, house monotone apportionment solution is a Quota method solution; indeed, it suffices to find just one example in which some Quota solution may be "twiddled" slightly (e.g., by interchanging the order in which some two states receive successive seats) while still satisfying house monotonicity and quota. Of course, such solutions will have a certain arbitrariness about them, and in

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particular will not be "consistent," thus violating an intrinsic idea of what is meant by a "method."

Nevertheless, it is interesting to ask *how far* some arbitrary, quotatone solution may deviate from a Quota method solution. Still [7] has given a characterization of all such solutions; here we shall give a simpler characterization that relates the class of quota, house monotone solutions to the Quota method and, at a further remove, to the Jefferson method, **J**.

2. The deck of cards. Given **p** we define the *Jefferson deck*, $\mathbf{D} = \{(i, a, p_i/a)\}$, as a sequence of "cards," each card bearing the *name* of a state i, a number of seats a, and the average district size p_i/a if a seats are apportioned to state i, stacked, in decreasing order, by the *values* p_i/a, $1 \leqslant i \leqslant s$, and $a \geqslant 1$ integer.

In the sequel we drop any redundant mention of the populations **p**.

Any house monotone method may be described in terms of **D** as follows. At house size $h = 0$, set $\mathbf{a} = \mathbf{0}$ and begin with the full deck $\mathbf{D} = \mathbf{D}(\mathbf{0}, 0)$. Given any apportionment **a** for h, an apportionment for $h + 1$ is found by withdrawing a card of form $(i, a_i + 1, p_i/(a_i + 1))$ from the remaining deck $\mathbf{D}(\mathbf{a}, h)$ and giving $a_i + 1$ seats to state i.

To say an apportionment **a** for h satisfies lower quota is equivalent to saying $a_i + 1 > p_i h/p$ or

$$p_i/(a_i + 1) < p/h, \tag{1}$$

while to say **a** satisfies upper quota is equivalent to saying $a_i - 1 < p_i h/p$ or (for $h > 0$ and $a_i \geqslant 1$)

$$p_i/(a_i - 1) > p/h. \tag{2}$$

In this paper $p_i/0$ will be interpreted as having value plus infinity. So, knowing the position of p/h relative to the Jefferson deck **D** determines the apportionments which satisfy quota at h.

3. The Jefferson method. The *Jefferson method* **J** [3], [4] may be described as follows:

(i) $f_i(0) = 0$, $1 \leqslant i \leqslant s$ and $\mathbf{D}(\mathbf{0}, 0) = \mathbf{D}$.

(ii) If $f_i(h) = a_i$, $1 \leqslant i \leqslant s$, is an apportionment for h and $\mathbf{D}(\mathbf{a}, h)$ the remaining deck, let k be the name of the state on the topmost card. Then remove that card and let $f_k(h + 1) = a_k + 1$ and $f_i(h + 1) = a_i$ for $i \neq k$.

Notice that the number of seats on the discarded card is precisely equal to $a_k + 1$. Huntington [5] described **J** and certain other methods in essentially this manner.

J is clearly house monotone. It also satisfies lower quota. For, suppose not: then there is some state j with $p_j/(a_j + 1) \geqslant p/h$ (see (1)). Thus $p_j/a_j > p/h$ and

$$\frac{p}{h} = \frac{\sum_i p_i}{\sum_i a_i} > \min_i \frac{p_i}{a_i} = \frac{p_l}{a_l},$$

implying $p_j/(a_j + 1) > p_l/a_l$. This is a contradiction, since then the card $(j, a_j + 1, p_j/(a_j + 1))$ would have been chosen before the card $(l, a_l, p_l/a_l)$.

It has been shown that house monotonicity and satisfying lower quota together with a "consistency" (see §8) property uniquely characterizes **J** [3].

4. The quota method. Let $U(\mathbf{a}, h)$ be the set of states which are eligible to receive an extra seat in a house of size $h + 1$ without violating upper quota, $U(\mathbf{a}, h) = \{i : p_i/a_i > p/(h + 1)\}$. These states can be ascertained by looking at the ordered deck of discarded cards.

The *Quota method* **Q** [1], [4] may be described as follows:

(i) $f_i(0) = 0$, $1 \leqslant i \leqslant s$ and $\mathbf{D}(\mathbf{0}, 0) = \mathbf{D}$.

(ii) If $f_i(h) = a_i$, $1 \leqslant i \leqslant s$, is an apportionment for h and $\mathbf{D}(\mathbf{a}, h)$ the remaining deck, let k be the name of the state on the topmost card bearing the name of a state in $U(\mathbf{a}, h)$. Then, remove that card and let $f_k(h + 1) = a_k + 1$ and $f_i(h + 1) = a_i$ for $i \neq k$.

It has been shown that house monotonicity and satisfying quota together with a weakened "consistency" property uniquely characterizes $\mathbf{Q}$ [4].

5. Quotatone methods. Let $\mathbf{D}$ be the Jefferson deck. For any house $h \geqslant 0$ and apportionment $\mathbf{a}$ for h let $\alpha(\mathbf{a}, h)$ be the *first* integer $\alpha \geqslant 1$ such that there are at least α cards in the remaining deck $\mathbf{D}(\mathbf{a}, h)$ with value $\geqslant p/(h + \alpha)$, and let $L(\mathbf{a}, h)$ be the set of all state names appearing on the first $\alpha(\mathbf{a}, h)$ cards. If no such α exists then set $\alpha = \infty$ and let $L(\mathbf{a}, h)$ be the set of all states (not all α need be checked, see below).

The meaning of $\alpha = \alpha(\mathbf{a}, h)$ is the following: if $\alpha < \infty$, $\mathbf{f}(\mathbf{p}, h) = \mathbf{a}$ is some apportionment at h, and the $(h + 1)$st seat is given to some state $k \notin L(\mathbf{a}, h)$, then there can be no monotone extension $\mathbf{g}$ of $\mathbf{f}^h$ such that $\mathbf{g}$ satisfies lower quota at house $h + \alpha$. The reason is that the allocation by $\mathbf{g}$ of each seat from $h + 1$ to $h + \alpha$ corresponds to the removal of a card from the remaining deck, so (by choice of α) at $h + \alpha$ there is still at least one card remaining corresponding to some state $j \in L(\mathbf{a}, h)$ and having value $p_j/b \geqslant p/(h + \alpha)$. Now, since at $h + \alpha$ state j has $b' < b$ seats we have

$$p_j/(b' + 1) \geqslant p_j/b \geqslant p/(h + \alpha),$$

which shows by (1) that state j violates lower quota at $h + \alpha$. Therefore if $\mathbf{f}$ is a quotatone apportionment solution, then $\mathbf{f}$ satisfies

(i) $\mathbf{f}(\mathbf{p}, 0) = \mathbf{0}$,

(ii) if $\mathbf{f}(\mathbf{p}, h) = \mathbf{a}$ and $f_k(\mathbf{p}, h + 1) = a_k + 1$, then $k \in L(\mathbf{a}, h) \cap U(\mathbf{a}, h)$.

The significance of $\alpha(\mathbf{a}, h)$ was to determine which states belonged to $L(\mathbf{a}, h)$. If there is no "first" α, then all states must belong to $L(\mathbf{a}, h)$. It is clearly unnecessary to inspect values of α larger than those which assure that every card of form $(i, a_i + 1, p_i/(a_i + 1))$ has value greater than or equal to $p/(h + \alpha)$. Define, then, β_i to be the least positive integer satisfying $p_i/(a_i + 1) \geqslant p/(h + \beta_i)$, that is $\beta_i = \lceil p(a_i + 1)/p_i - h \rceil$ and $\beta = \max_i \beta_i$. Then $L(\mathbf{a}, h)$ may be defined as before with this modification: if there is no first integer $\beta > \alpha \geqslant 1$ for which at least α cards in $\mathbf{D}(\mathbf{a}, h)$ have value $\geqslant p/(h + \alpha)$, then let $L(\mathbf{a}, h)$ be the set of all states.

Let $\overline{\mathbf{Q}}$ be the class of all solutions $\mathbf{f}$ satisfying (i) and (ii).

THEOREM 1. *$\overline{\mathbf{Q}}$ is presisely the set of all quotatone solutions.*

PROOF. We know that every quotatone solution is in $\overline{\mathbf{Q}}$, hence $\mathbf{Q} \subseteq \overline{\mathbf{Q}}$. Further, every $\mathbf{f} \in \overline{\mathbf{Q}}$ satisfies upper quota, by definition. Suppose $\mathbf{f}(\mathbf{p}, h + 1) = \mathbf{b}$ violates lower quota at $h + 1$ for state k. Then $p_k/(b_k + 1) \geqslant p/(h + 1)$ by (1), and card $d_0 = (k, b_k + 1, p_k/(b_k + 1))$ is in $\mathbf{D}(\mathbf{b}, h + 1)$ hence also in $\mathbf{D}(\mathbf{f}(h), h)$. Therefore $\alpha(\mathbf{f}(h), h) = 1$, hence the $(h + 1)$st card removed, d_{h+1}, also had value greater than or equal to $p/(h + 1)$, and $d_{h+1} \neq d_0$. Hence $\alpha(f(h - 1), h - 1) \leqslant 2$ so d_h(the hth card removed) had value greater than or equal to $p/(h + 1)$. In general, $\alpha(\mathbf{f}(h'), h') + h' \leqslant h + 1$ for $h' \leqslant h$, and in $\mathbf{D}$ there were $h + 2$ cards, $d_0, d_1, \ldots, d_{h+1}$ with values $\geqslant p/(h + 1)$. But then no house monotone solution can satisfy quota at $h + 1$, contradicting the fact that $\overline{\mathbf{Q}} \supseteq \mathbf{Q} \neq \emptyset$. ∎

Thus every quotatone solution is a variant of a Quota method solution in the following sense: instead of giving the additional seat at each successive house to the *first* state satisfying upper quota, it is given to some state satisfying upper quota among the *first* α states. The problem is to decide *which* of the α states to select: the selection of the first one (satisfying upper quota) turns out to be the only resolution that is "consistent" (see §8) subject to satisfying quota.

6. Generalized quota. It is necessary to generalize definitions, methods and theorems to the need for any admissible apportionment $\mathbf{a}$ to satisfy certain minimum *requirements* $\mathbf{r} = (r_1, \ldots, r_s)$, where the integer $r_i \geqslant 0$ is the minimum number of seats which must be accorded to state i by mandate.[1] Letting $h^* = \sum_i r_i$, an *apportionment* for $h \geqslant h^*$ is an n-tuple of integers $\mathbf{a} = (a_i, \ldots, a_s)$, with $\mathbf{a} \geqslant \mathbf{r}$ and $\sum_i a_i = h$. A *solution* is a function $\mathbf{f}(\mathbf{p}, \mathbf{r}, h)$ which every $\mathbf{p}$ and $\mathbf{r}$, and all $h \geqslant h^* = \sum_i r_i$ associates a unique apportionment for h, $a_i = f_i(\mathbf{p}, \mathbf{r}, h) \geqslant r_i$, where $\sum_i a_i = h$. The concepts *method*, *extension*, *solution up to h* are defined analogously to the pure ($\mathbf{r} = 0$) case.

It is impossible, for certain values $\mathbf{r}$, to ask for solutions satisfying quota. Thus, this definition must be modified. The motivation is this. Suppose the exact quota of state i at h is less than or equal to r_i: then it deserves no more than r_i seats, but is required to receive at least r_i seats. Therefore, we reason, it should receive exactly r_i seats, and we say its lower and upper quota should be exactly r_i. Eliminate these states whose apportionment is fixed, and subtract the corresponding r_i's from $h = h_0$ to obtain h_1 seats which must be distributed among the remaining states. Using this smaller house h_1, compute exact quotas, that is, compute the proportional share of h_1 that each of the remaining states deserves, and iterate.

Specifically, let $J_0 = J_0(h) = \{1, \ldots, s\}$, $h_0 = h(\geqslant h^*)$ and define $J_1 = J_1(h) = \{i \in J_0;\, p_i h_0/\sum_{J_0} p_j > r_i\}$ and $h_1 = h_0 - \sum_{i \notin J_1} r_i$. In general, $J_{\beta+1} = J_{\beta+1}(h) = \{i \in J_\beta;\, p_i h_\beta/\sum_{J_\beta} p_j > r_i\}$ and $h_{\beta+1} = h_0 - \sum_{i \notin J_{\beta+1}} r_i < h_\beta$, the process stopping with J_μ when $J_{\mu+1} = J_\mu$. Thus, $J_0(h) \supset J_1(h) \supset \cdots \supset J_\mu(h) = J(h)$ and $h = h_0 > h_1 > \cdots > h_\mu$, with $p_i h_\mu/\sum_{J_\mu} p_j > r_i$ for $i \in J_\mu = J$. We define the (*generalized*) *exact quota* $q_i(\mathbf{p}, \mathbf{r}, h)$ of state i to be

$$
\begin{aligned}
q_i(\mathbf{p}, \mathbf{r}, h) &= r_i \quad \text{for } i \notin J(h), \\
&= p_i h_\mu \Big/ \sum_{J(h)} p_j \quad \text{for } i \in J(h).
\end{aligned}
$$

Thus, the (*generalized*) *lower quota* of state i is $l_i(h) = \lfloor q_i(\mathbf{p}, \mathbf{r}, h) \rfloor$ and the (*generalized*) *upper quota* is $u_i(h) = \lceil q_i(\mathbf{p}, \mathbf{r}, h) \rceil$. This means, in particular, that $l_i(h) = u_i(h) = r_i$ for $i \notin J(h)$. Note that this definition is slightly more natural than given previously in [4] and simplifies the proof of Theorem 4 in that paper. This altered definition of upper quota can result in different apportionments than those given by the Quota method with minimum requirements as previously defined [4]. It has been pointed out by Mayberry [6] that this occurs for our 1984B example [4].

There is a more direct way of computing $J(h)$. By definition,

$$p_i h_\beta \Big/ \sum_{J_\beta} p_j \leqslant r_i \quad \text{for } i \in J_\beta \sim J_{\beta+1},\ 0 \leqslant \beta \leqslant \mu. \tag{3}$$

Therefore,

$$h_{\beta+1} = h_\beta - \sum_{J_\beta \sim J_{\beta+1}} r_i \leqslant h_\beta - \frac{h_\beta \sum_{J_\beta \sim J_{\beta+1}} p_i}{\sum_{J_\beta} p_j} = \frac{h_\beta \sum_{J_{\beta+1}} p_j}{\sum_{J_\beta} p_j},$$

and so

$$\sum_{J_{\beta+1}} p_j / h_{\beta+1} \geqslant \sum_{J_\beta} p_j / h_\beta$$

or

$$\sum_{J_{\beta+1}} p_j \Big/ \left(h - h^* + \sum_{J_{\beta+1}} r_j \right) \geqslant \sum_{J_\beta} p_j \Big/ \left(h - h^* + \sum_{J_\beta} r_j \right).$$

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[1] In the United States $r_i = 1$ for all i; in France $r_i = 2$ for all i; in the recently reformed European Parliament the r_i's vary between 6 and 36.

From this and (3) we deduce

$$p_i/r_i > \frac{\sum\limits_{J(h)} p_j}{h - h^* + \sum\limits_{J(h)} r_j} \geqslant p_k/r_k \quad \text{for all } i \in J(h),\, k \notin J(h). \tag{4}$$

But (4) uniquely determines $J(h)$ by the following procedure. Suppose, for simplicity, $p_1/r_1 \geqslant p_2/r_2 \geqslant \cdots \geqslant p_s/r_s$. Given h and h^* consider, $\lambda_1 = p_1/(h - h^* + r_1)$. If $\lambda_1 \geqslant p_2/r_2$, stop, $J(h) = \{1\}$. Otherwise, consider $\lambda_2 = (p_1 + p_2)/(h - h^* + r_1 + r_2) > \lambda_1$. If $\lambda_2 \geqslant p_3/r_3$, stop, $J(h) = \{1, 2\}$. Otherwise, continue similarly.

Define $U(\mathbf{a}, h)$ to be the set of states eligible to receive an extra seat in a house of size $h + 1$ without violating (generalized) upper quota, $U(\mathbf{a}, h) = \{i : a_i + 1 \leqslant u_i(h + 1)\}$. Then the (generalized) *Quota method* $\mathbf{Q}(\mathbf{r})$ [1], [3] with requirements is exactly the same as $\mathbf{Q}$ except that $f_i(\mathbf{p}, \mathbf{r}, h^*) = r_i$ for all i, and $\mathbf{D}(\mathbf{a}, h^*)$ is the original deck $\mathbf{D}$ from which has been eliminated all cards $(i, a, p_i/a)$ with $a \leqslant r_i$ for all i. Again, note that $\mathbf{Q}(\mathbf{r})$ is defined differently here than in [4] because of the altered definition of generalized exact, upper and lower quota.

It is interesting to note that whenever pure lower quota *can* be satisfied, the Quota method does so.

THEOREM 2. *If there exists an apportionment at h satisfying pure lower quota then there exists a* $\mathbf{Q}$-*apportionment which does so.*

PROOF. Suppose that there exists an apportionment at some $h'(\geqslant h^* = \sum_i r_i)$ which satisfies pure lower quotas. Then, surely, $\sum_i \max\{\lfloor p_i h'/p \rfloor, r_i\} \leqslant h'$, where $p = \sum_i p_i$.

Let $p = \sum_i p_i$. Suppose $\mathbf{a}$ is a $\mathbf{Q}$-apportionment which does not satisfy $a_i \geqslant \lfloor p_i h'/p \rfloor$ for all i. Then there exists j such that $a_j < \lfloor p_j h'/p \rfloor$, whence $p_j/(a_j + 1) \geqslant p/h'$, and therefore there must be some l with $a_l > p_l h'/p$.

Let $L = \{k;\, a_k > p_k h'/p\} \neq \emptyset$, and $R = \{i;\, r_i \geqslant p_i h'/p\}$. For any $i \notin L$ we have $a_i \leqslant \lfloor p_i h'/p \rfloor$; in particular, $a_j < \lfloor p_j h'/p \rfloor$. Further, if we assume $L \subseteq R$ then

$$\sum_{i \in L} a_i + \sum_{i \notin L} a_i \leqslant \sum_{i \in L} r_i + \sum_{i \notin L} a_i < \sum_{i \in L} r_i + \sum_{i \notin L} \lfloor p_i h'/p \rfloor \leqslant h',$$

contradicting the fact that $\mathbf{a}$ is an apportionment for h'. Thus, there exists $l \in L \sim R$, that is, a state for which $a_l > p_l h'/p > r_l$, implying

$$p_l/a_l < p/h' \leqslant p_j/(a_j + 1).$$

Let h_l be the house at which state l received its a_lth seat; and choose $l \in L \sim R$ such that h_l is largest. Clearly $h_l < h'$, since state j is eligible to receive an extra seat at h'. Let

$$K = \{i;\ \text{state } i \text{ received a seat at } h,\ h_l < h \leqslant h_0\}.$$

By choice of l, $l \notin K$.

Suppose $k \in K$ and $a_k > p_k h'/p$, then $k \in L$ and $k \notin R$, but $h_k > h_l$, a contradiction. Therefore, $k \in K$ implies

$$f_k(h') = a_k \leqslant p_k h'/p. \tag{5}$$

However, $f_k(h_l) < a_k$ for $k \in K$ and so

$$p_k/(f_k(h_l) + 1) \geqslant p_k/a_k \geqslant p/h' > p_l/a_l = p_l/f_l(h_l)$$

showing that $k \in K$ is ineligible at h_l, that is,

$$f_k(h_l) = f_k(h_l - 1) \geqslant p_k h_l/p \quad \text{for } k \in K. \tag{6}$$

But, in the interval $h_l < h \leqslant h'$ exactly $h' - h_l$ seats were awarded to states in K, so $\sum_K \{f_k(h') - f_k(h_l)\} = h' - h_l$. Subtracting (6) from (5), then summing over K

$$h' - h_l = \sum_K \{f_k(h') - f_k(h_l)\} \leqslant \sum_K (p_k/p)(h' - h_l)$$

implying, since $h' - h_l > 0$, that $\sum_K p_k/p \geqslant 1$, a contradiction since $l \notin K$. This completes the proof.

7. Generalized quotatone methods. Given minimum requirements $(r_1, \ldots, r_s) = \mathbf{r}$, let $h^* = \sum_i r_i$, and for any $h > h^*$ let $l_i(h)$, $u_i(h)$ be the generalized lower and upper quotas for state i, and $J(h)$ the set of "slack" states. Then an apportionment $\mathbf{a}$ for h *satisfies (generalized) quota* if and only if for each i, $u_i(h) \geqslant a_i \geqslant l_i(h)$, and a solution $\mathbf{f}$ *satisfies quota* if all its apportionments do. We note that, for $i \notin J(h)$, $a_i \geqslant l_i(h)$ is equivalent to

$$a_i + 1 > p_i\Big(h - h^* + \sum_{J(h)} r_j\Big)\Big/ \sum_{J(h)} p_j,$$

or

$$p_i/(a_i + 1) < \sum_{J(h)} p_j \Big/ \Big((h - h^*) + \sum_{J(h)} r_j\Big). \tag{7}$$

For any given $h \geqslant 0$ and any apportionment $\mathbf{a}$ for h consider the Jefferson deck $\mathbf{D}(\mathbf{a}, h)$ remaining after all cards $(i, a_i', p_i/a_i')$, $0 \leqslant a_i' \leqslant a_i$, are removed. Define $\alpha = \alpha(\mathbf{a}, h)$ to be the first integer $\alpha \geqslant 1$ such that there are at least α distinct cards in $\mathbf{D}(\mathbf{a}, h)$ having value $\geqslant \sum_{J(h+\alpha)} p_j/((h - h^*) + \sum_{J(h+\alpha)} r_j + \alpha)$, and let $L(\mathbf{a}, h)$ be the set of all state names appearing on the first α cards. If no such α exists then set $\alpha = \infty$ and let $L(\mathbf{a}, h)$ be the set of all states (see below).

Suppose that $\mathbf{f}(\mathbf{p}, h) = \mathbf{a}$ is some apportionment at h, and the $(h + 1)$st seat is given to some state $k \notin L(\mathbf{a}, h)$. Then $\alpha < \infty$ and in constructing a house monotone extension of $\mathbf{f}^h$ exactly α cards must be removed in going from $h + 1$ to $h + \alpha$. By choice of α there remains at $h + \alpha$ some card $(j, b_j, p_j/b_j)$ with value

$$p_j/b_j \geqslant \sum_{J(h+\alpha)} p_j \Big/ \Big((h - h^*) + \sum_{J(h+\alpha)} r_j + \alpha\Big),$$

and since state j has fewer than b_j seats at $h + \alpha$, it follows from (7) that lower quota at $h + \alpha$ is violated. Hence if $\mathbf{f}$ is a house monotone apportionment solution satisfying quota (for the given requirements $\mathbf{r}$), then we must have

(i) $\mathbf{f}(h^*) = \mathbf{r}$;

and

(ii) if $\mathbf{f}(h) = \mathbf{a}$ and $f_k(h + 1) = a_k + 1$,

then

$$k \in L(\mathbf{a}, h) \cap U(\mathbf{a}, h).$$

At this point we note that, in the definition of $\alpha = \alpha(\mathbf{a}, h)$ it is unneccesary to inspect values of α larger than those which assure that the cards $(i, a_i + 1, p_i/(a_i + 1))$ have value

$$p_i/(a_i + 1) \geqslant \sum_{J(h+1)} p_j \Big/ \Big((h - h^*) + \sum_{J(h+1)} r_j + \alpha\Big) \quad \text{for each } i,$$

$1 \leqslant i \leqslant s$, since for any $\alpha \geqslant 1$, $J(h + 1) \subseteq J(h + \alpha)$ and by (4)

$$\sum_{J(h+1)} p_i \Big/ \Big((h - h^*) + \sum_{J(h+1)} r_j + \alpha\Big) \geqslant \sum_{J(h+\alpha)} p_i \Big/ \Big((h - h^*) + \sum_{J(h+\alpha)} r_j + \alpha\Big).$$

Hence if we define

$$\beta = \max_i \left\lceil \left\{ (a_i + 1) \sum_{J(h+1)} p_j \right\} \Big/ p_i - \sum_{J(h+1)} r_j - (h - h^*) \right\rceil,$$

then $L(\mathbf{a}, h)$ may be defined as above with the modification: if there is no first integer α in the range $\beta > \alpha \geqslant 1$ satisfying the condition, let $L(\mathbf{a}, h)$ be the set of all states.

Let $\bar{\mathbf{Q}}(\mathbf{r})$ be the class of all solutions $\mathbf{f}$ satisfying (i) and (ii).

THEOREM 3. $\bar{\mathbf{Q}}(\mathbf{r})$ *is precisely the set of all quotatone solutions for the requirements* $\mathbf{r}$.

The proof parallels that of theorem 1.

8. Concluding remarks. Two fundamental properties of apportionment methods are dictated by common sense and firmly grounded in the history of the problem: house monotonicity and satisfying quota.

Following the idea of Still that the class of all methods having these two properties are in some sense describable, we have shown that in fact they may all be described by using the Jefferson deck and choosing a card "near the top" that satisfies upper quota. If minimum requirements are given, we have shown that the quota idea has a natural generalization, and that the class of all house monotone (generalized) quota methods are again describable in a natural way in terms of the Jefferson deck.

Nevertheless, since there are a multiplicity of apportionment methods which satisfy the above properties, the problem remains: which among these methods should be used? Here a third principle comes into play, which has its basis in the pioneering work on apportionment methods by E. V. Huntington in the early part of this century [5], and touches on the idea of what is meant intuitively by "method." Briefly stated, if for some problem and **M**-solution $\mathbf{f}$ one state, having population $\bar{p}$ and $\bar{a}$ seats at house h gets the "next" (i.e., $(h + 1)$st) seat before another state having population p^* and a^* seats, then the first state has *priority* over the second state, written $(\bar{p}, \bar{a}) \gtrsim (p^*, a^*)$. If in another problem we also have $(p^*, a^*) \gtrsim (\bar{p}, \bar{a})$, then we say the states are *tied*, written $(p^*, a^*) \sim (\bar{p}, \bar{a})$. The method **M** is said to be *consistent* if it treats tied states equally with respect to receiving one more seat; that is, there must be an alternate **M**-solution $\mathbf{f}'$ which is an extension of $\mathbf{f}^h$ and gives the $(h + 1)$st seat instead to the p^*-state. The essence of the idea is that a "method" should not *change* priorities between a pair of states if the data of some other state populations are altered.

The five methods proposed by Huntington, as well as their generalizations [2], all have this property. Moreover it may be shown that every house monotone, consistent method is necessarily a Huntington method, that is, uses a "rank index" $r(p, a)$ which tells which state (with population p and number of seats a) most deserves to receive one more seat. Specifically,

(i) $\mathbf{f}(0) = 0$,

and

(ii) if $\mathbf{f}(h) = \mathbf{a}$ and k is some one state maximizing $r(p_k, a_k)$, then

$$f_k(h + 1) = a_k + 1, \qquad f_i(h + 1) = a_i \quad \text{all } i \neq k.$$

The desirable features of these methods are, first, they are eminently computable, and second, they are based on the natural idea of comparing the states *pairwise* to determine which is worst off, hence most deserving of an extra seat. On the other hand, none of the Huntington methods satisfies quota [4].

Given the precedent—in the political context—of the two principles, house monotonicity and satisfying quota, it is natural to ask whether there is some modification of the consistency concept that leads to a computationally simple method. Indeed there is: let consistency be modified to apply between pairs of states only when both states *are eligible* (i.e., both are in $U(\mathbf{a}, h)$). Then the Quota method is the *unique* method that is quota, house monotone, and consistent in this weaker sense [4]. (Note that with the present definition of generalized upper quotas, the restriction in [4] to unbiased requirements is unnecessary.) Moreover it is clear from the preceding that the Quota method $\mathbf{Q}$ is the computationally simplest and most natural within the class $\overline{\mathbf{Q}}$. If the concept is weakened still further to apply only between pairs of states that are both eligible *and* among the first $\alpha(\mathbf{a}, h)$ states, that is in $L(\mathbf{a}, h) \cap U(\mathbf{a}, h)$, then we may expect that this property, together with house-monotonicity and satisfying quota, determine precisely the class of methods defined as follows.

Let $r(p, a)$ be a rank index and let $\mathbf{r}$ be a given set of minimum requirements, $h^* = \sum_i r_i$. Define the *quotatone* method $\mathbf{M}$ *based on* $r(p, a)$ to be the set of all solutions $\mathbf{f}$ obtained as follows. For any $\mathbf{r}$,

(i) $\mathbf{f}(h^*) = \mathbf{r}$,

(ii) if $\mathbf{f}(h) = \mathbf{a}$ and k is some one state that maximizes $r(p_k, a_k)$ over all $k \in L(\mathbf{a}, h) \cap U(\mathbf{a}, h)$ then let

$$f_k(h+1) = a_k + 1, \qquad f_i(h+1) = a_i \quad \text{all } i \neq k.$$

Thus, for example, $\mathbf{Q}$ is the quotatone method based on $r(p, a) = p/(a+1)$.

Among the class of all such methods $\mathbf{Q}$ is the simplest and most natural, since it does not depend on the computation of $L(\mathbf{a}, h)$, which in general is complex. Furthermore, although computers make possible the calculation of quotatone apportionments for any rank index, it is nevertheless of paramount importance that political men both understand and feel comfortable with any method that is used. It may be that the set $L(\mathbf{a}, h)$ is simply beyond political understanding.

There are several criteria which are clearly of primary importance in choosing an apportionment method: satisfying quota, house monotonicity, consistency, and "simplicity." The desiderata cannot be met simultaneously. The question is to find a satisfactory reconciliation. Consistency and house monotonicity determine Huntington methods, which are simple but do not satisfy quota. A slightly weakened consistency notion together with satisfying quota gives the Quota method, which has an intuitive simplicity. A considerably weakened consistency idea leads to quotatone methods based on some $r(p, a)$ which, however, lack this intuitive simplicity.

Note added in proof. J. W. Still has pointed out that theorem 2 applies to our first definition of generalized quotas ([1], [4]) and not to the altered definition introduced in this paper.

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