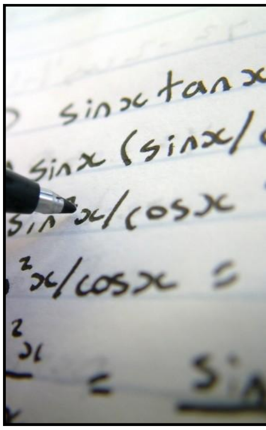


Introduction to informatics

Jerzy Nawrocki
Faculty of Computing & Inf. Sci.
Poznan University of Technology
jerzy.nawrocki@put.poznan.pl

Numerical Methods

Introduction to informatics

A problematic computation

$$\sqrt{a^2 + b^2} = a\sqrt{1 + (b/a)^2} = b\sqrt{1 + (a/b)^2}$$

```
begin
a:= 3e-25; b:= 4e-25;
m:= sqrt(a*a + b*b);
writeln(m)
end.
```

```
begin
a:= 3e-25; b:= 4e-25;
if a > b then
  m:= a*sqrt(1+ (b/a)*(b/a))
else
  m:= b*sqrt(1+ (a/b)*(a/b));
writeln(m)
end.
```

0.0000000000E+00 ≠ 5.0000000000E-25

Numerical methods (2)

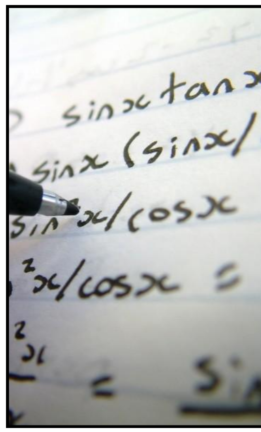
Introduction to informatics

Aim

Present:

- Classical representation of real numbers
- How to compute standard functions (e^x, cos x, π)
- How to compute value of a polynomial

Numerical methods (3)



Introduction to informatics

Agenda

- **Representation of real numbers**
- **Newton method and √x**
- **Maclaurin series and functions e^x, cos x**
- **Polynomials and Horner scheme**
- **Ill-conditioned problems**

Numerical methods (4)

Introduction to informatics

Representation of real numbers

0	0	1	0	1	1	0	0

Integer part — Fraction part

Numerical methods (5)

Introduction to informatics

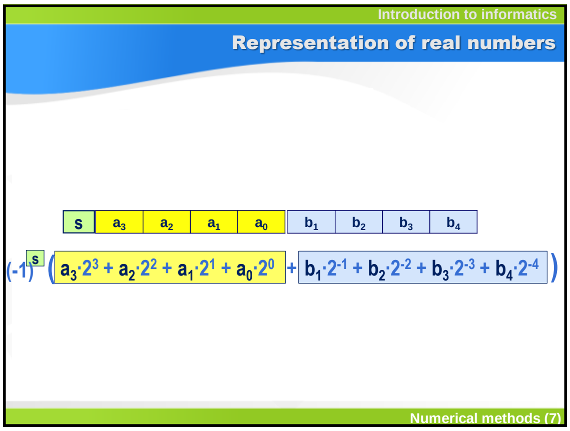
Representation of real numbers

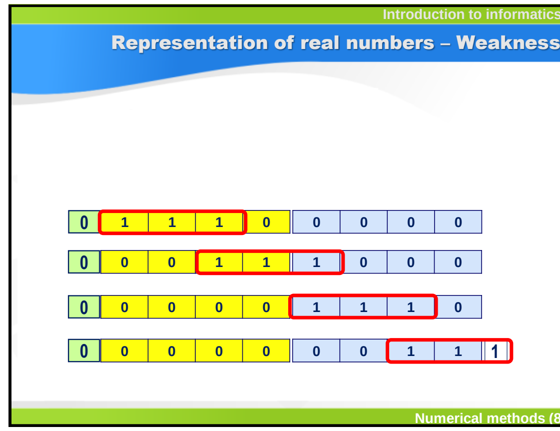
0	0	1	0	1	1	0	0

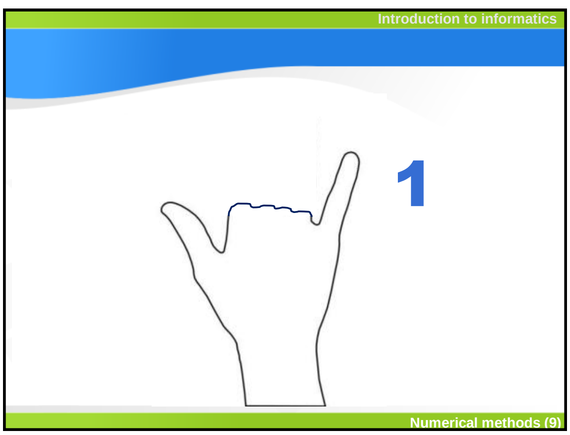
Value = $0\cdot2^3 + 0\cdot2^2 + 1\cdot2^1 + 0\cdot2^0$ + $1\cdot2^{-1} + 1\cdot2^{-2} + 0\cdot2^{-3} + 0\cdot2^{-4}$

= 2 + 0,75

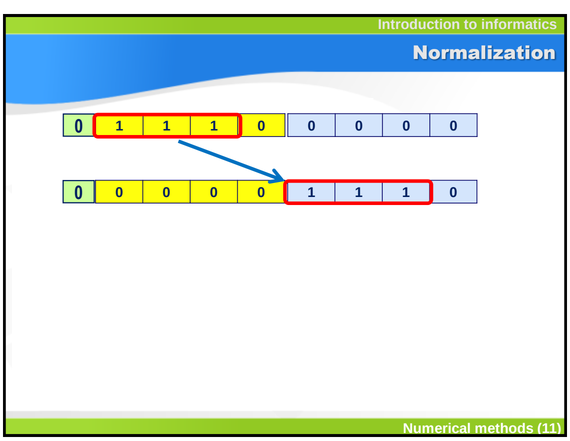
Numerical methods (6)

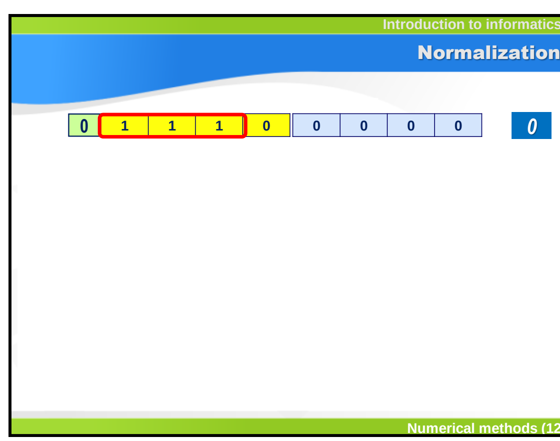

Introduction to informatics
Representation of real numbers
s a3 a2 a1 a0 b1 b2 b3 b4
$(-1)^{s}\left(a_3 \cdot 2^3 + a_2 \cdot 2^2 + a_1 \cdot 2^1 + a_0 \cdot 2^0 + b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + b_3 \cdot 2^{-3} + b_4 \cdot 2^{-4}\right)$
Numerical methods (7)

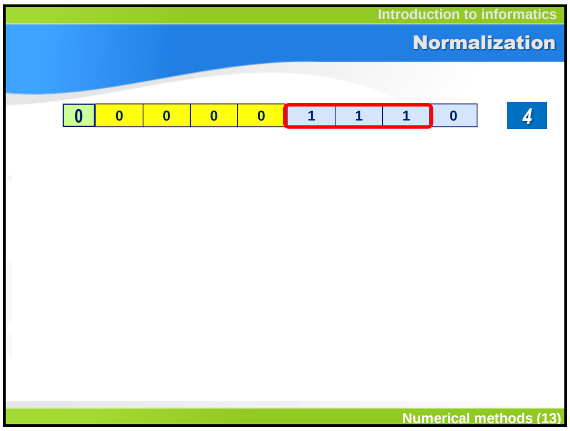

Introduction to informatics
Representation of real numbers – Weakness
0 1 1 1 0 0 0 0 0
0 0 0 1 1 1 0 0 0
0 0 0 0 0 1 1 1 0
0 0 0 0 0 0 0 1 1 1
Numerical methods (8)

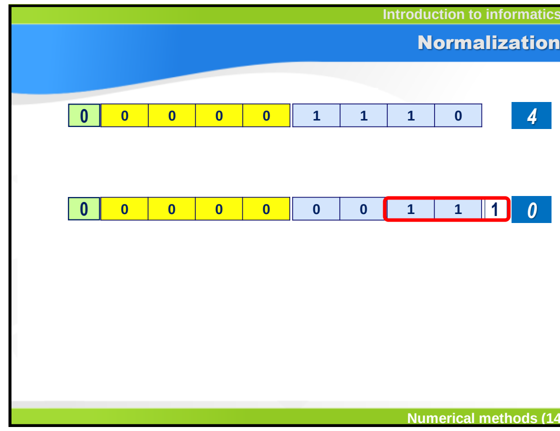

Introduction to informatics
1
Numerical methods (9)

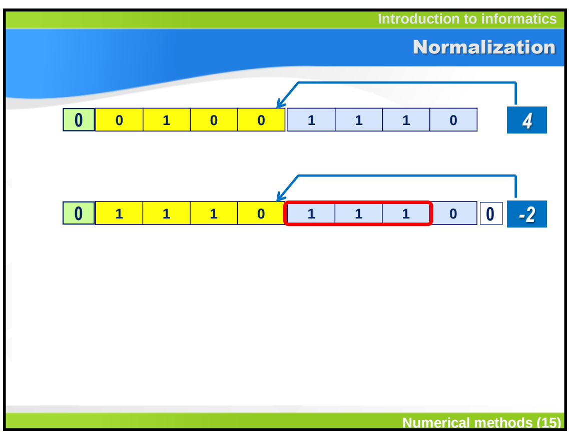

Introduction to informatics
7
Numerical methods (10)

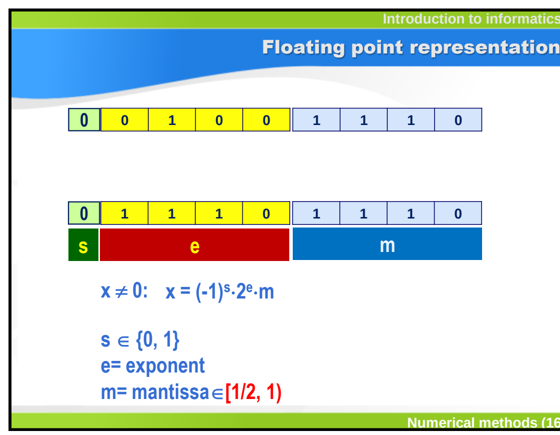

Introduction to informatics
Normalization
0 1 1 1 0 0 0 0 0
0 0 0 0 0 1 1 1 0
Numerical methods (11)

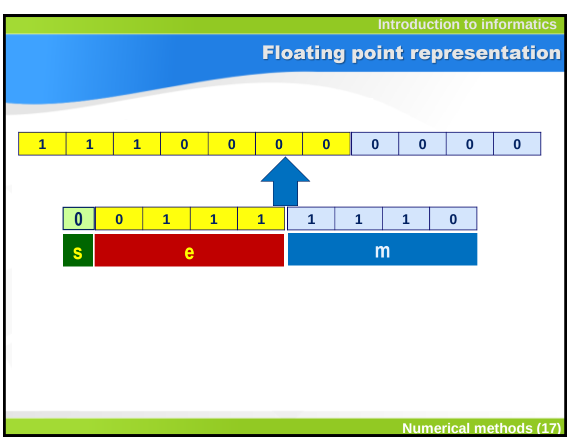

Introduction to informatics
Normalization
0 1 1 1 0 0 0 0 0
0
Numerical methods (12)

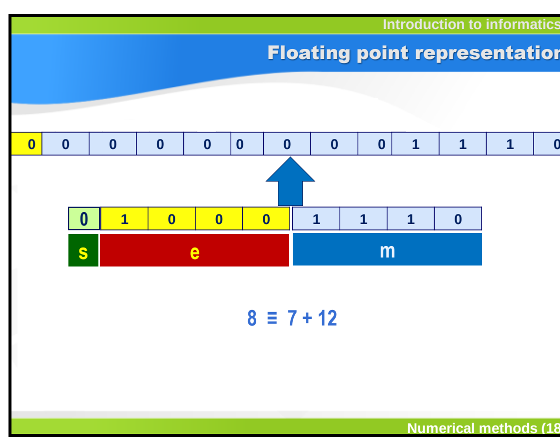

Introduction to informatics
Normalization
0 0 0 0 0 1 1 1 0
4
Numerical methods (13)


Introduction to informatics
Normalization
0 0 0 0 0 1 1 1 0
4
0 0 0 0 0 0 0 1 1 1 0
0
Numerical methods (14)

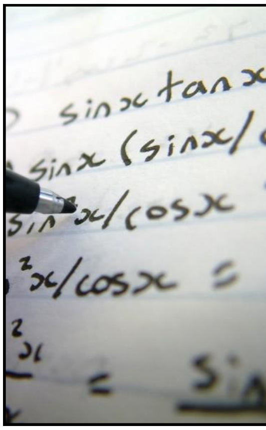

Introduction to informatics
Normalization
0 0 1 0 0 1 1 1 0
4
0 1 1 1 0 1 1 1 0 0
-2
Numerical methods (15)


Introduction to informatics
Floating point representation
0 0 1 0 0 1 1 1 0
0 1 1 1 0 1 1 1 0
s
e
m
x ≠ 0: x = (-1)^s·2^e·m
s ∈ {0, 1}
e= exponent
m= mantissa ∈ [1/2, 1)
Numerical methods (16)


Introduction to informatics
Floating point representation
1 1 1 0 0 0 0 0 0 0 0
0 0 1 1 1 1 1 1 0
s
e
m
Numerical methods (17)


Introduction to informatics
Floating point representation
0 0 0 0 0 0 0 0 0 1 1 1 0
0 1 0 0 0 1 1 1 0
s
e
m
8 ≡ 7 + 12
Numerical methods (18)

Agenda



- Representation of real numbers
- **Newton method and √x**
- Maclaurin series and functions e^x, cos x
- Polynomials and Horner scheme
- Ill-conditioned problems

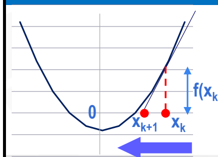
Square root

$g(a) = \sqrt{a}$

Transformation to finding zeroes of f(x)

$f(x) = x^2 - a = 0$

Newton method:



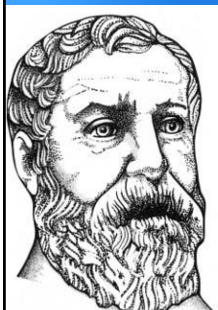
Geom. interpretation of derivative:

$f'(x_k) = f(x_k) / (x_k - x_{k+1})$

where $f'(x) = 2x$. After rewriting:

$x_{k+1} = \frac{1}{2} (x_k + a/x_k)$

Iterative algorithm



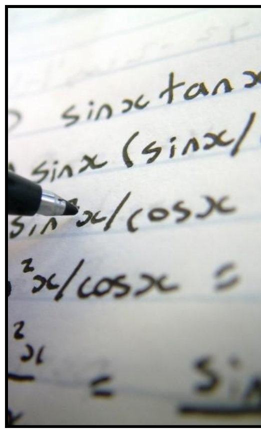
Heron of Alexandria
From a German translation of *Pneumatica*, 1688 r.
http://pl.wikipedia.org/wiki/Heron_z_Aleksandrii

Heron's algorithm of computing a square root of *a*

$$x_1 = \begin{cases} a & \text{if } a >= 1 \\ 1 & \text{if } a < 1 \end{cases}$$

$x_{k+1} = \frac{1}{2} (x_k + a/x_k)$

Agenda



- Representation of real numbers
- Newton method and √x
- **Maclaurin series and functions e^x, cos x**
- Polynomials and Horner scheme
- Ill-conditioned problems

Taylora series

Brook Taylor
1685 - 1731

1703 – 1709: Cambridge University

1712: Royal Society

1715: Taylor series (without proof)

$$f(x) \doteq \sum_{k=0}^{N} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}.$$

Maclaurin series

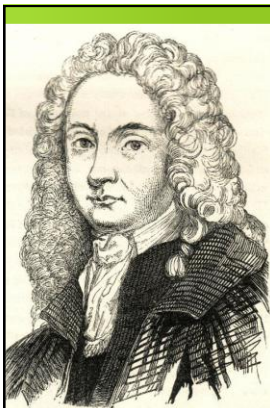
Colin Maclaurin (1698 – 1746)

$$f(x) \doteq \sum_{k=0}^{N} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}.$$

For $x_0 = 0$ we have:

$$f(x) \doteq \sum_{k=0}^{N} \frac{f^{(k)}(0)x^k}{k!}$$

Maclaurin series


Colin Maclaurin (1698 – 1746)

$$f(x) \cong \sum_{k=0}^{N} \frac{f^{(k)}(0)\, x^k}{k!}$$

f(x) = e^x

- $(e^x)' = e^x$
- $e^0 = 1$

$$e^x \approx \sum_{k=0}^{N} x^k / k!$$

$e^x \cong x^0/0! + x^1/1! + x^2/2! + x^3/3! + ..$

$= 1 + x/1! + x^2/2! + x^3/3! + ..$

e^x

$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + ...$

e(1) = 2,71..
e(0) = 1

$T = \frac{num}{den}$ → $\frac{1}{1}$, $\frac{x}{1!}$, $\frac{x^2}{2!}$, $\frac{x^3}{3!}$

e^x

$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + ...$

e(1) = 2,71..
e(0) = 1

$T = \frac{num}{den}$: $\frac{1}{1}$ →(*x, *1) $\frac{x}{1!}$ →(*x, *2) $\frac{x^2}{2!}$ →(*x, *3) $\frac{x^3}{3!}$

e^x

$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + ...$

e(1) = 2,71..
e(0) = 1

```
void main(){
  float x;          // Argument e(x)
  scanf("%g", &x);
  printf("e(%g)= %g\n", x, e(x));
  return;
  }
```


Maclaurin series


Colin Maclaurin (1698 – 1746)

$$f(x) \cong \sum_{k=0}^{N} \frac{f^{(k)}(0)\, x^k}{k!}$$

f(x) = cos x

- $(\cos x)^{(0)} = \cos x$ — 1
- $(\cos x)' = -\sin x$ — 0
- $(\cos x)'' = -\cos x$ — -1
- $(\cos x)^{(3)} = \sin x$ — 0
- $(\cos x)^{(4)} = \cos x$ — 1
- . . .

$\cos x \approx 1 + 0 - x^2/2! + 0 + x^4/4! + ..$

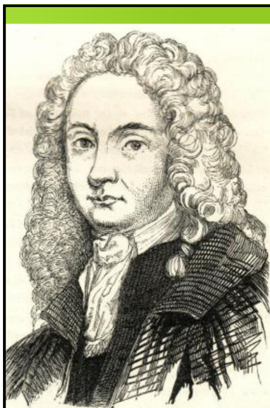
cos(x)

$\cos x \approx 1 + 0 - x^2/2! + 0 + x^4/4! + ..$

cos(0) = 1
cos(1,57..) = 0

$T = \frac{num}{den}$: $\frac{1}{1}$ →(*(-x^2), *1*2) $\frac{-x^2}{2!}$ →(*(-x^2), *3*4) $\frac{x^4}{4!}$

π



$$f(x) \cong \sum_{k=0}^{N} \frac{f^{(k)}(0)x^k}{k!}$$

tg(π/4)= ???

arctg(1) = π/4

$$\text{arctg}(x) = \sum_{k=0} \frac{(-1)^k}{2k+1} x^{2k+1}$$

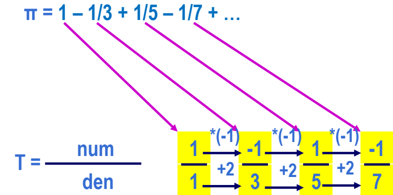
$\pi/4 = 1 - x^3/3 + x^5/5 - x^7/7 + \ldots$

$= 1 - 1/3 + 1/5 - 1/7 + \ldots$

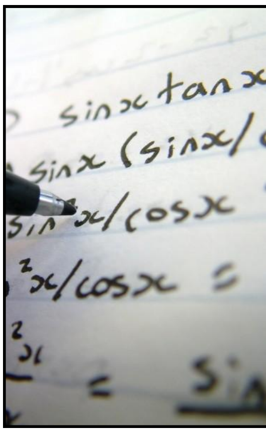
Colin Maclaurin (1698 – 1746)

π

$\pi = 1 - 1/3 + 1/5 - 1/7 + \ldots$



Agenda



- Representation of real numbers
- Newton method and √x
- Maclaurin series and functions e^x, cos x
- **Polynomials and Horner scheme**
- Ill-conditioned problems

Polynomials

$$p(x) = \sum_{k=0} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$$

Signature of p

Call of p

```
float p(float x, int n, float a[ ])
  ...
#define MaxN 50
void main(){
  float x;              // Zmienna x
  int   n;              // Stopien wielomianu
  float a[MaxN+1]; // Wspolczynniki wielomianu
  ...
  printf("%g\n", p(x, n, a));
  return;
  }
```


Polynomials

$$p(x) = \sum_{k=0} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$$

```
float p(float x, int n, float a[ ]){
    float result, PowerX;
    int k = 0;
    PowerX = 1;
    result = a[0] * PowerX;
    while (k < n){
        k = k+1;
        PowerX = PowerX * x;
        result = result + a[k] * PowerX;
        }
    return result;
    }
```


Horner scheme

$p(x,n) = a_0x^n + a_1x^{n-1} + .. + a_{n-1}x^1 + a_n$

$p(x,0) = a_0$

$p(x,1) = a_0x + a_1$

$p(x,2) = a_0x^2 + a_1x + a_2$

$p(x,3) = a_0x^3 + a_1x^2 + a_2x + a_3$

$p(x,4) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$

Horner scheme

$p(x,n) = a_0x^n + a_1x^{n-1} + .. + a_{n-1}x^1 + a_n$

$p(x,0) = a_0$

$p(x,1) = a_0x + a_1 = p(x, 0)x + a_1$

$p(x,2) = a_0x^2 + a_1x + a_2$

$p(x,3) = a_0x^3 + a_1x^2 + a_2x + a_3$

$p(x,4) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$

Horner scheme

$p(x,n) = a_0x^n + a_1x^{n-1} + .. + a_{n-1}x^1 + a_n$

$p(x,0) = a_0$

$p(x,1) = a_0x + a_1 = p(x, 0)x + a_1$

$p(x,2) = a_0x^2 + a_1x + a_2 = (a_0x^1 + a_1)x + a_2 = p(x,1)x + a_2$

$p(x,3) = a_0x^3 + a_1x^2 + a_2x + a_3$

$p(x,4) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$

Horner scheme

$p(x,n) = a_0x^n + a_1x^{n-1} + .. + a_{n-1}x^1 + a_n$

$p(x,0) = a_0$

$p(x,1) = a_0x + a_1 = p(x, 0)x + a_1$

$p(x,2) = a_0x^2 + a_1x + a_2 = (a_0x^1 + a_1)x + a_2 = p(x,1)x + a_2$

$p(x,3) = a_0x^3 + a_1x^2 + a_2x + a_3 =$

$= (a_0x^2 + a_1x^1 + a_2)x + a_3 = p(x,2)x + a_3$

$p(x,4) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 =$

$= (a_0x^3 + a_1x^2 + a_2x^1 + a_3)x + a_4 = p(x,3)x + a_4$

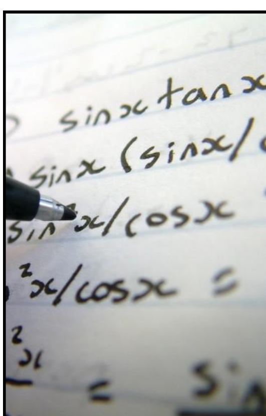
$p(x,n) = p(x, n-1)x + a_n$

Horner scheme

$p(x,n) = a_0x^n + a_1x^{n-1} + .. + a_{n-1}x^1 + a_n$

$p(x,0) = a_0$

$p(x,n) = p(x, n-1)x + a_n$



Agenda

- Representation of real numbers
- Newton method and √x
- Maclaurin series and functions e^x, cos x
- Polynomials and Horner scheme
- **Ill-conditioned problems**

Ill-conditioned problems

A problem is ill-conditioned if:

A small relative error in the data → much larger relative error in the result(s)

$$p(x) = a_{20}\,x^{20} + \dots + a_1\,x + a_0 = \prod_{k=1}^{20}(x-k) = 0$$

$x = 1, 2, \dots, 20$

$a_{19} = -210 \rightarrow a_{19} = -(210 + 2^{-23})$

$x = 15 \rightarrow x = 13{,}99 + 2{,}5i$