THE FIRST APPROACH TO THE INTERVAL GENERALIZED FINITE DIFFERENCES

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Outline

- Conventional generalized finite differences
- Interval generalized finite differences
- Numerical examples
- Discussion of results and final conclusions

Literature overview

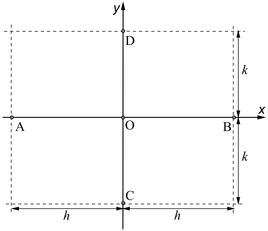
- **Jensen, P.S.**, Finite difference techniques for variable grids, *Computers & Structures* **2** (1-2), 17–29 **(1972)**
- **Orkisz, J.**: Computer approach to the finite difference method (in Polish). *Computer and Mechanics* **2**, 7–69 **(1979)**
- **Orkisz, J.**: Meshless finite difference method I – Basic approach. Meshless finite difference method II – Adaptative approach. In: Idelsohn, S.R., Oñate, E.N., Dvorkin, E. (eds.) *Computational Mechanics*. New Trends and Applications. CIMNE **(1998)**
- **Liszka, T., Orkisz, J.**: The finite difference method at arbitrary irregular grids and its application in applied mechanics. *Computers & Structures* **11** (1–2), 83–95 **(1980)**
- **Benito, J.J., Urena, F., Gavete, L.**: Solving parabolic and hyperbolic equations by the generalized finite difference method. *Journal of Computational and Applied Mathematics* **209** (2), 208–233 **(2007)**
- **Urena, F., Salete, E., Benito, J.J., Gavete, L.**: Solving third- and fourth-order partial differential equations using GFDM: application to solve problems of plates. *International Journal of Computer Mathematics* **89** (3), 366–376 **(2012)**

Conventional generalized finite differences

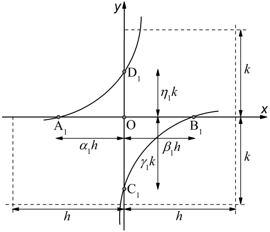
In the area of **conventional finite differences** we can indicate two main classes of methods. They differ, in particular, in the way the points of a grid are located in the domain:

- **the classical finite differences (FD)**
  - a regular grid of points is generated (the distances between two neighbouring points in a given direction are equal); otherwise, we have to apply different formulas of classical FD for these points;
  - the formulas are derived on the bases of the Taylor series expansion;
  - an approximation of one particular derivative at a given point is obtained;
- **the generalized finite differences (GFD)**
  - an arbitrary (irregular) arrangement of points in the region is allowed, although the regular distribution can be also applied;
  - the formulas are derived on the bases of the Taylor series expansion;
  - an approximation of a complete set of derivatives up to the order n is obtained at one time.

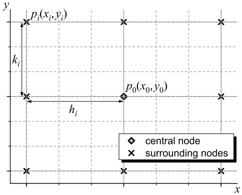
Examples of regular and irregular grids



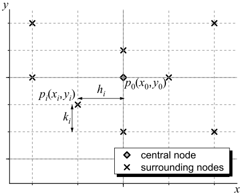
(a) regular 5-point grid



(b) irregular 5-point grid



(c) regular 9-point grid



(d) irregular 9-point grid

Figure

Conventional generalized finite differences

Consider the derivatives of a function $u = u(x, y)$

$$\frac{\partial u}{\partial x}(p_0)\,, \quad \frac{\partial u}{\partial y}(p_0)\,, \quad \frac{\partial^2 u}{\partial x^2}(p_0)\,, \quad \frac{\partial^2 u}{\partial y^2}(p_0)\,, \quad \frac{\partial^2 u}{\partial x \partial y}(p_0)\,. \tag{1}$$

We assume that the function u has continuous derivatives up to the third order with respect to x and y in a region $\Omega \subset \mathbb{R}^2$.

The values of these derivatives at some point p_0 can be computed with generalized finite differences as described in detail in [Benito, et al.][1].

[1] **Benito, J.J., Urena, F., Gavete, L.**: Solving parabolic and hyperbolic equations by the generalized finite difference method. *Journal of Computational and Applied Mathematics* **209** (2), 208–233 **(2007)**

Conventional generalized finite differences

We generate a grid (cloud) of points such that the point $p_0 = (x_0, y_0)$ is **the central node** and the points $p_i = (x_i, y_i)$, $i = 1, 2, \ldots n$ are **the i-th nodes** of the star. We have $h_i = x_i - x_0$, $k_i = y_i - y_0$.

We expand the function u in the Taylor series about the point p_0 and evaluate it at the points p_i, $i = 1, 2, \ldots n$. For each point p_i we have

$$\begin{aligned} u(p_i) &= u(p_0) + h_i \frac{\partial u}{\partial x}(p_0) + k_i \frac{\partial u}{\partial y}(p_0) \\ &+ \frac{1}{2!}\left(h_i^2 \frac{\partial^2 u}{\partial x^2}(p_0) + k_i^2 \frac{\partial^2 u}{\partial y^2}(p_0) + 2h_i k_i \frac{\partial^2 u}{\partial x \partial y}(p_0) \right) \\ &+ \frac{1}{3!}\left(h_i^3 \frac{\partial^3 u}{\partial x^3}(q_i) + k_i^3 \frac{\partial^3 u}{\partial y^3}(q_i) + 3h_i^2 k_i \frac{\partial^3 u}{\partial x^2 \partial y}(q_i) + 3h_i k_i^2 \frac{\partial^3 u}{\partial x \partial y^2}(q_i) \right), \end{aligned} \tag{2}$$

where $q_i = (\xi_i, \eta_i)$ is **an intermediate point of the remainder term** such that $\xi_i \in \left(\xi_i^{\min}, \xi_i^{\max}\right)$, $\eta_i \in \left(\eta_i^{\min}, \eta_i^{\max}\right)$. Furthermore, we have $\xi_i^{\min} = \min\{x_i, x_0\}$, $\xi_i^{\max} = \max\{x_i, x_0\}$ and $\eta_i^{\min} = \min\{y_i, y_0\}$, $\eta_i^{\max} = \max\{y_i, y_0\}$.

Conventional generalized finite differences

If we add the above expressions, we obtain

$$\sum_{i=1}^{N}\left(u\left(p_i\right)-u\left(p_0\right)\right)=\sum_{i=1}^{N}h_i\frac{\partial u}{\partial x}\left(p_0\right)+\sum_{i=1}^{N}k_i\frac{\partial u}{\partial y}\left(p_0\right)$$
$$+\frac{1}{2}\left(\sum_{i=1}^{N}h_i^2\frac{\partial^2 u}{\partial x^2}\left(p_0\right)+\sum_{i=1}^{N}k_i^2\frac{\partial^2 u}{\partial y^2}\left(p_0\right)+2\sum_{i=1}^{N}h_ik_i\frac{\partial^2 u}{\partial x\partial y}\left(p_0\right)\right) \qquad (3)$$
$$+\frac{1}{6}\sum_{i=1}^{N}r\left(q_i\right),$$

where

$$r\left(q_i\right)=h_i^3\frac{\partial^3 u}{\partial x^3}\left(q_i\right)+k_i^3\frac{\partial^3 u}{\partial y^3}\left(q_i\right)+3h_i^2k_i\frac{\partial^3 u}{\partial x^2\partial y}\left(q_i\right)+3h_ik_i^2\frac{\partial^3 u}{\partial x\partial y^2}\left(q_i\right). \qquad (4)$$

Conventional generalized finite differences

We define the function $\mathcal{F}$ as follows

$$\begin{aligned}\mathcal{F}(u) = \sum_{i=1}^{N} \Bigg\{ \Bigg[& u(p_0) - u(p_i) + h_i \frac{\partial u}{\partial x}(p_0) + k_i \frac{\partial u}{\partial y}(p_0) + \frac{1}{2} h_i^2 \frac{\partial^2 u}{\partial x^2}(p_0) \\ & + \frac{1}{2} k_i^2 \frac{\partial^2 u}{\partial y^2}(p_0) + h_i k_i \frac{\partial^2 u}{\partial x \partial y}(p_0) + \frac{1}{6} r(q_i) \Bigg] w(h_i, k_i) \Bigg\}^2,\end{aligned} \tag{5}$$

where $w = w(h_i, k_i)$ are the weight functions simply denoted by w_i. We minimize $\mathcal{F}$ with respect to the values of the derivatives at the point p_0. We have

$$\frac{\partial \mathcal{F}(u)}{\partial \mathcal{A}} = \frac{\partial \mathcal{F}(u)}{\partial \mathcal{B}} = \frac{\partial \mathcal{F}(u)}{\partial \mathcal{C}} = \frac{\partial \mathcal{F}(u)}{\partial \mathcal{D}} = \frac{\partial \mathcal{F}(u)}{\partial \mathcal{E}} = 0, \tag{6}$$

where

$$\mathcal{A} = \frac{\partial u}{\partial x}(p_0), \quad \mathcal{B} = \frac{\partial u}{\partial y}(p_0), \quad \mathcal{C} = \frac{\partial^2 u}{\partial x^2}(p_0), \quad \mathcal{D} = \frac{\partial^2 u}{\partial y^2}(p_0), \quad \mathcal{E} = \frac{\partial^2 u}{\partial x \partial y}(p_0). \tag{7}$$

Conventional generalized finite differences

Finally, we obtain a linear system of equations of the form

$$\widehat{A}\widehat{D} = \widehat{B} + \widehat{E}, \tag{8}$$

where

$$\widehat{A} = \begin{bmatrix} \sum_{i=1}^{N} h_i^2 w_i^2 & \sum_{i=1}^{N} h_i k_i w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i^3 w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i k_i^2 w_i^2 & \sum_{i=1}^{N} h_i^2 k_i w_i^2 \\ \sum_{i=1}^{N} h_i k_i w_i^2 & \sum_{i=1}^{N} k_i^2 w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i^2 k_i w_i^2 & \sum_{i=1}^{N} \frac{1}{2} k_i^3 w_i^2 & \sum_{i=1}^{N} h_i k_i^2 w_i^2 \\ \sum_{i=1}^{N} \frac{1}{2} h_i^3 w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i^2 k_i w_i^2 & \sum_{i=1}^{N} \frac{1}{4} h_i^4 w_i^2 & \sum_{i=1}^{N} \frac{1}{4} h_i^2 k_i^2 w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i^3 k_i w_i^2 \\ \sum_{i=1}^{N} \frac{1}{2} h_i k_i^2 w_i^2 & \sum_{i=1}^{N} \frac{1}{2} k_i^3 w_i^2 & \sum_{i=1}^{N} \frac{1}{4} h_i^2 k_i^2 w_i^2 & \sum_{i=1}^{N} \frac{1}{4} k_i^4 w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i k_i^3 w_i^2 \\ \sum_{i=1}^{N} h_i^2 k_i w_i^2 & \sum_{i=1}^{N} h_i k_i^2 w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i^3 k_i w_i^2 & \sum_{i=1}^{N} \frac{1}{2} h_i k_i^3 w_i^2 & \sum_{i=1}^{N} h_i^2 k_i^2 w_i^2 \end{bmatrix}, \tag{9}$$

$$\widehat{D} = \left[\frac{\partial u}{\partial x}(p_0), \frac{\partial u}{\partial y}(p_0), \frac{\partial^2 u}{\partial x^2}(p_0), \frac{\partial^2 u}{\partial y^2}(p_0), \frac{\partial^2 u}{\partial x \partial y}(p_0) \right]^{\mathrm{T}}, \tag{10}$$

Conventional generalized finite differences

$$\widehat{B} = \begin{bmatrix} \sum_{i=1}^{N} \left(-u\left(p_0\right) + u\left(p_i\right)\right) h_i w_i^2 \\ \sum_{i=1}^{N} \left(-u\left(p_0\right) + u\left(p_i\right)\right) k_i w_i^2 \\ \sum_{i=1}^{N} \left(-u\left(p_0\right) + u\left(p_i\right)\right) \frac{1}{2} h_i^2 w_i^2 \\ \sum_{i=1}^{N} \left(-u\left(p_0\right) + u\left(p_i\right)\right) \frac{1}{2} k_i^2 w_i^2 \\ \sum_{i=1}^{N} \left(-u\left(p_0\right) + u\left(p_i\right)\right) k_i h_i w_i^2 \end{bmatrix}, \quad \widehat{E} = \begin{bmatrix} -\sum_{i=1}^{N} r\left(q_i\right) h_i w_i^2 \\ -\sum_{i=1}^{N} r\left(q_i\right) k_i w_i^2 \\ -\sum_{i=1}^{N} r\left(q_i\right) \frac{1}{2} h_i^2 w_i^2 \\ -\sum_{i=1}^{N} r\left(q_i\right) \frac{1}{2} k_i^2 w_i^2 \\ -\sum_{i=1}^{N} r\left(q_i\right) k_i h_i w_i^2 \end{bmatrix}. \tag{11}$$

Note that as the weight functions w_i we choose as in [Benito, et al.][1]
$w\left(h_i, k_i\right) = 1/d_i^3$, where $d_i = \left(\left(x_0 - x_i\right)^2 + \left(y_0 - y_i\right)^2\right)^{1/2} = \left(h_i^2 + k_i^2\right)^{1/2}$.

[1] **Benito, J.J., Urena, F., Gavete, L.**: Solving parabolic and hyperbolic equations by the generalized finite difference method. *Journal of Computational and Applied Mathematics* **209** (2), 208–233 **(2007)**

Conventional generalized finite differences

Remark 1.

Let u_0, u_i, $i = 1, 2, \dots n$, approximate the exact values $u(p_0)$, $u(p_i)$ of the function u at the central and surrounding nodes. If we also ignore the remaining terms of the Taylor series expansion given in the components of a vector $\widehat{E}$, **we obtain the linear system of equations whose solution provides approximate values of a complete set of the first and second order derivatives of u at the central node p_0**.

Such an approach utilizes the **conventional generalized finite differences**.

The matrix $\widehat{A}$ of the linear system of equations (8) is *symmetrical*. As proposed in, e.g., [Benito, et al.][1], this system of equations can be efficiently solved with **the Cholesky method**.

[1] **Benito, J.J., Urena, F., Gavete, L.**: Solving parabolic and hyperbolic equations by the generalized finite difference method. *Journal of Computational and Applied Mathematics* **209** (2), 208–233 **(2007)**

Interval generalized finite differences

PRELIMINARY ASSUMPTIONS

- $X_i,\ Y_i,\ i = 0, 1, \ldots, n$, denote the intervals such that $x_i \in X_i,\ y_i \in Y_i$;
- $U = U(X, Y)$ denotes the interval extension of $u = u(x, y)$;
- $H_i = X_i - X_0,\ K_i = Y_i - Y_0$;
- $W(H_i, K_i) = 1/D_i^3$, where $D_i = (H_i^2 + K_i^2)^{1/2}$.

ASSUMPTIONS ABOUT VALUES OF THE DERIVATIVES IN THE MIDPOINTS

- $D^{(3,1)} = D^{(3,1)}(X, Y)$, $D^{(3,2)} = D^{(3,2)}(X, Y)$, $D^{(3,3)} = D^{(3,3)}(X, Y)$ and $D^{(3,4)} = D^{(3,4)}(X, Y)$ denote the interval extensions of the derivatives of u, i.e., $\partial^3 u/\partial x^3\,(x, y)$, $\partial^3 u/\partial y^3\,(x, y)$, $\partial^3 u/\partial x^2 \partial y\,(x, y)$ and $\partial^3 u/\partial x \partial y^2\,(x, y)$;
- for the midpoints ξ_i, η_i, we assume that there exist the intervals such that $\xi_i \in \Xi_i,\ \eta_i \in \mathrm{H}_i,\ i = 1, 2, \ldots, n$.

Interval generalized finite differences

Based on the above assumptions we define the interval extension $R = R(X, Y)$ of the error term function $r = r(x, y)$ and compute its value at the point (Ξ_i, H_i).

We have

$$R_i = H_i^3 D_i^{(3,1)} + K_i^3 D_i^{(3,2)} + 3H_i^2 K_i D_i^{(3,3)} + 3H_i K_i^2 D_i^{(3,4)}, \tag{12}$$

where $R_i = R(\Xi_i, \mathrm{H}_i)$, $D_i^{(3,j)} = D^{(3,j)}(\Xi_i, \mathrm{H}_i)$, $i = 1, 2, \ldots, n$, $j = 1, 2, 3, 4$.

If we replace all real values used in (8)-(11) by the appropriate intervals and all functions by their interval extensions, then we obtain an interval linear system of equations of the form

$$AD = B + E, \tag{13}$$

Interval generalized finite differences

where the matrix A is given as

$$\begin{bmatrix} \sum_{i=1}^{N} H_i^2 W_i^2 & \sum_{i=1}^{N} H_i K_i W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i^3 W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i K_i^2 W_i^2 & \sum_{i=1}^{N} H_i^2 K_i W_i^2 \\ \sum_{i=1}^{N} H_i K_i W_i^2 & \sum_{i=1}^{N} K_i^2 W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i^2 K_i W_i^2 & \sum_{i=1}^{N} \frac{1}{2} K_i^3 W_i^2 & \sum_{i=1}^{N} H_i K_i^2 W_i^2 \\ \sum_{i=1}^{N} \frac{1}{2} H_i^3 W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i^2 K_i w_i^2 & \sum_{i=1}^{N} \frac{1}{4} H_i^4 W_i^2 & \sum_{i=1}^{N} \frac{1}{4} H_i^2 K_i^2 W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i^3 K_i W_i^2 \\ \sum_{i=1}^{N} \frac{1}{2} H_i K_i^2 W_i^2 & \sum_{i=1}^{N} \frac{1}{2} K_i^3 W_i^2 & \sum_{i=1}^{N} \frac{1}{4} H_i^2 K_i^2 W_i^2 & \sum_{i=1}^{N} \frac{1}{4} K_i^4 W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i K_i^3 W_i^2 \\ \sum_{i=1}^{N} H_i^2 K_i W_i^2 & \sum_{i=1}^{N} H_i K_i^2 W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i^3 K_i W_i^2 & \sum_{i=1}^{N} \frac{1}{2} H_i K_i^3 W_i^2 & \sum_{i=1}^{N} H_i^2 K_i^2 W_i^2 \end{bmatrix}, \tag{14}$$

$$D = \left[D_0^{(X)},\ D_0^{(Y)},\ D_0^{(XX)},\ D_0^{(YY)},\ D_0^{(XY)} \right]^{\mathrm{T}}, \tag{15}$$

Interval generalized finite differences

$$B = \begin{bmatrix} \sum_{i=1}^{N} (-U_0 + U_i) H_i W_i^2 \\ \sum_{i=1}^{N} (-U_0 + U_i) K_i W_i^2 \\ \sum_{i=1}^{N} (-U_0 + U_i) \frac{1}{2} H_i^2 W_i^2 \\ \sum_{i=1}^{N} (-U_0 + U_i) \frac{1}{2} K_i^2 W_i^2 \\ \sum_{i=1}^{N} (-U_0 + U_i) K_i H_i W_i^2 \end{bmatrix}, \quad E = \begin{bmatrix} -\sum_{i=1}^{N} R_i H_i W_i^2 \\ -\sum_{i=1}^{N} R_i K_i W_i^2 \\ -\sum_{i=1}^{N} R_i \frac{1}{2} H_i^2 W_i^2 \\ -\sum_{i=1}^{N} R_i \frac{1}{2} K_i^2 W_i^2 \\ -\sum_{i=1}^{N} R_i K_i H_i W_i^2 \end{bmatrix}. \tag{16}$$

Interval generalized finite differences

Remark 2.

We can solve the interval linear system of equations (13) with the interval Cholesky method. Such a choice has an important consequence that can be easily noticed when we follow the theorem provided by [Moore, et al.][1].

Consider a finite system of linear algebraic equations of the form $Ax = b$, where A is an n-by-n matrix, b is an n-dimensional vector and the coefficients of A and b are real or interval values. The existence of the solution to $Ax = b$ is provided by the following theorem.

Theorem (Moore, et al.[1])

If we can carry out all the steps of a direct method for solving $Ax = b$ in the interval arithmetic (if no attempted division by an interval containing zero occurs, nor any overflow or underflow), then the system has a unique solution for every real matrix in A and every real matrix in b, and the solution is contained in the resulting interval vector X.

[1] **Moore, R.E., Kearfott, R.B., Cloud, M.J.**: *Introduction to Interval Analysis*. SIAM Philadelphia **(2009)**

Interval generalized finite differences

Remark 3.

In the interval approach proposed, all real value coefficients of the matrix $\widehat{A}$ and the vectors $\widehat{B}$, $\widehat{E}$ of the linear system of equations (8) are included in the interval value coefficients of the matrix A and the vectors B, E of the interval linear system of equations (13).

Hence, based on the Theorem 1 we can conclude as follows.

If we solve the interval linear system of equations (13) with the interval Cholesky method (i.e., the interval counterpart of the direct Cholesky method), then the exact values of the derivatives given in D (10) at the node p_0 are included in the interval values of the vector $\widehat{D}$ (15) and we have

$$\begin{gathered}\frac{\partial u}{\partial x}(p_0) \in D_0^{(X)}, \;\; \frac{\partial u}{\partial y}(p_0) \in D_0^{(Y)}, \\ \frac{\partial^2 u}{\partial x^2}(p_0) \in D_0^{(XX)}, \;\; \frac{\partial^2 u}{\partial y^2}(p_0) \in D_0^{(YY)}, \;\; \frac{\partial^2 u}{\partial x \partial y}(p_0) \in D_0^{(XY)}.\end{gathered} \tag{17}$$

Numericals examples

TEST FUNCTIONS

We consider the following functions

$$
\begin{aligned}
u_1(x,y) &= \exp(xy), & u_2(x,y) &= \left(x^2+y^2+0.5\right)^2 \exp(xy), \\
u_3(x,y) &= \cos(x)\cos(y), & u_4(x,y) &= u_a(x,y)+u_b(x,y),
\end{aligned}
\tag{18}
$$

where the functions u_a, u_b are given in the form

$$
u_a(x,y) = \frac{3}{4}\exp\left(-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4}\right)+\frac{3}{4}\exp\left(-\frac{(9x+1)^2}{49}-\frac{(9y+1)^2}{10}\right),
$$

$$
u_b(x,y) = \frac{1}{2}\exp\left(-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4}\right)-\frac{1}{5}\exp\left(-(9x-4)^2-(9y-7)^2\right).
$$

Numericals examples

ASSUMPTIONS & SETTINGS

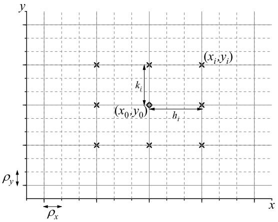
- the regular/irregular grid;
- the number of points in the star: 9, 17, 25;
- the aim is to compute the derivatives of the functions: u_1, u_2, u_3 and u_4;
- the distances ρ_x, ρ_y that are further used to determine the position of the points p_0, p_i, $i = 1, 2, \ldots, n$ in the star (used to examine the influence of the distances between the points in the cloud).

THE AIM OF NUMERICAL EXPERIMENTS

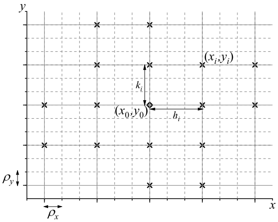
A computation of the interval values of the second order derivatives of the functions u_1, u_2, u_3 and u_4 at the point $p_0 = (0.5; 0.5)$.

- Example 1. Numerical results in the case of the exact bounds of the error term coefficient for the functions u_1, u_3 and u_4.
- Example 2. Numerical results in the case of both the exact and approximated bounds of the error term coefficient for the function u_2.

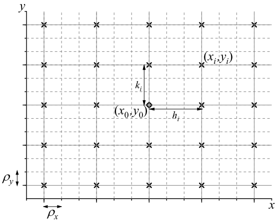
Examples of regular and irregular grids of n-nodes



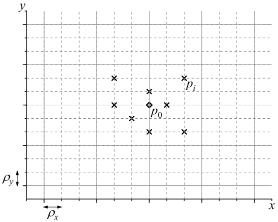
(a) regular 9-point grid



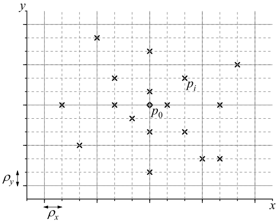
(b) regular 17-point grid



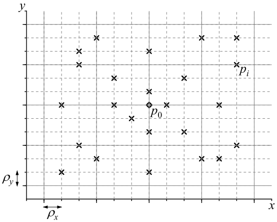
(c) regular 25-point grid



(d) irregular 9-point grid

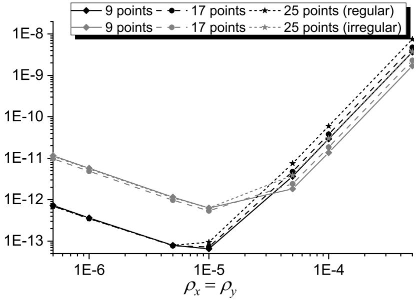


(e) irregular 17-point grid

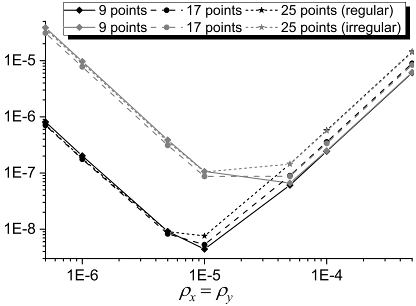


(f) irregular 25-point grid

Example 1. Numerical results for the function u_1



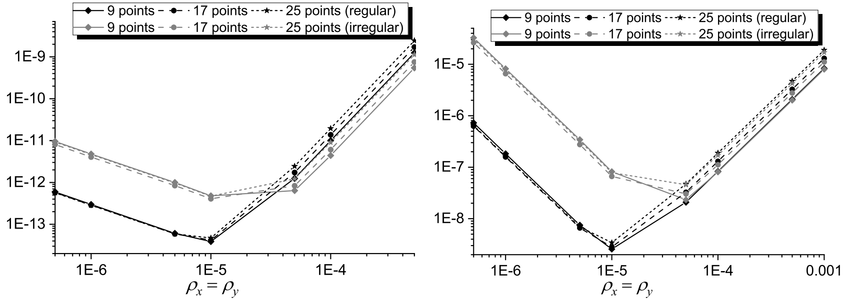
(a) $\partial u_1(p_0)/\partial x$



(b) $\partial^2 u_1(p_0)/\partial x^2$

Figure: Widths of interval enclosures of the derivatives $\partial u(p_0)/\partial x$ and $\partial^2 u(p_0)/\partial x^2$ of the functions u_1 for different values of the grid parameter $\rho_x = \rho_y$.

Example 1. Numerical results for the function u_3

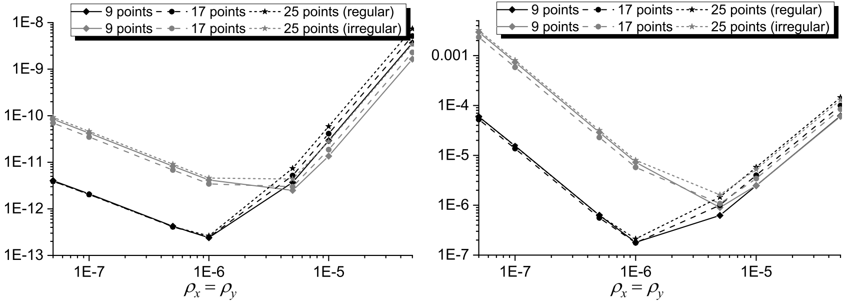


(a) $\partial u_3(p_0)/\partial x$

(b) $\partial^2 u_3(p_0)/\partial x^2$

Figure: Widths of interval enclosures of the derivatives $\partial u(p_0)/\partial x$ and $\partial^2 u(p_0)/\partial x^2$ of the functions u_3 for different values of the grid parameter $\rho_x = \rho_y$.

Example 1. Numerical results for the function u_4



(a) $\partial u_4(p_0)/\partial x$ (b) $\partial^2 u_4(p_0)/\partial x^2$

Figure: Widths of interval enclosures of the derivatives $\partial u(p_0)/\partial x$ and $\partial^2 u(p_0)/\partial x^2$ of the functions u_4 for different values of the grid parameter $\rho_x = \rho_y$.

Example 2. Approximated bounds of the error term coefficient

Let us compute the derivatives of u_2 at the point $p_0(0.5, 0.5)$ using a method of approximation of the error term intervals.

In the formula

$$R_i = H_i^3 D_i^{(3,1)} + K_i^3 D_i^{(3,2)} + 3H_i^2 K_i D_i^{(3,3)} + 3H_i K_i^2 D_i^{(3,4)},$$

we assume that we know the interval enclosures of the third order derivatives such that for a given point q_i the following relations hold

$$\frac{\partial^3 u}{\partial x^3}(q_i) \in D_i^{(3,1)} = \left[\underline{D}_i^{(3,1)}, \overline{D}_i^{(3,1)}\right], \ \frac{\partial^3 u}{\partial y^3}(q_i) \in D_i^{(3,2)} = \left[\underline{D}_i^{(3,2)}, \overline{D}_i^{(3,2)}\right],$$

$$\frac{\partial^3 u}{\partial x^2 \partial y}(q_i) \in D_i^{(3,3)} = \left[\underline{D}_i^{(3,3)}, \overline{D}_i^{(3,3)}\right], \ \frac{\partial^3 u}{\partial x \partial y^2}(q_i) \in D_i^{(3,4)} = \left[\underline{D}_i^{(3,4)}, \overline{D}_i^{(3,4)}\right].$$

Example 2. Approximated bounds of the error term coefficient

If the analytical formulas of the third order derivatives are not known, we have to approximate the endpoints of the error term intervals.

One approach assumes that we compute the derivatives up to the third order using the conventional generalized finite differences of higher order (see, e.g., [Urena, et al.][1]) and then we use the results obtained to approximate the endpoints considered. For $k = 1, 2, 3, 4$, we choose

$$\underline{D}_i^{(3,k)} \approx \min\left\{D_i^{(3,k)*}, D_0^{(3,k)*}\right\}, \; \underline{D}_i^{(3,k)} \approx \max\left\{D_i^{(3,k)*}, D_0^{(3,k)*}\right\}, \qquad (19)$$

where, for $s = i$ and $s = 0$, we take

$$D_s^{(3,1)*} = \frac{\partial^3 u(p_s)}{\partial x^3}, \; D_s^{(3,2)*} = \frac{\partial^3 u(p_s)}{\partial y^3}, \; D_s^{(3,3)*} = \frac{\partial^3 u(p_s)}{\partial x^2 \partial y}, \; D_s^{(3,4)*} = \frac{\partial^3 u(p_s)}{\partial x \partial y^2}.$$

[1] **Urena, F., Salete, E., Benito, J.J., Gavete, L.**: Solving third- and fourth-order partial differential equations using GFDM: application to solve problems of plates. *International Journal of Computer Mathematics* **89**(3), 366–376 **(2012)**

Example 2. Function u_2. Numerical results in the case of the exact bounds of the error term coefficient

Table: Exact values of the derivatives, their interval enclosures obtained with the analytical formula of u_2 and the widths of intervals (the regular 25-point grid with $\rho_x = \rho_y = 5\mathrm{E}{-6}$)

Deriv.	Interval enclosure of the derivative	Width
$\partial u/\partial x$	[3.21006354171905646E+0000, 3.21006354171965611E+0000] exact ≈ 3.21006354171935371E+0000	5.9963E−13
$\partial u/\partial y$	[3.21006354171905646E+0000, 3.21006354171965611E+0000] exact ≈ 3.21006354171935371E+0000	5.9963E−13
$\partial^2 u/\partial x^2$	[1.05932096408550928E+0001, 1.05932097243747383E+0001] exact ≈ 1.05932096876738672E+0001	8.3519E−08
$\partial^2 u/\partial y^2$	[1.05932096408551582E+0001, 1.05932097243747452E+0001] exact ≈ 1.05932096876738672E+0001	8.3519E−08
$\partial^2 u/\partial x \partial y$	[6.74113338979118222E+0000, 6.74113346974061664E+0000] exact ≈ 6.74113343761064279E+0000	7.9949E−08

Example 2. Function u_2. Numerical results in the case of the approximated bounds of the error term coefficient

Table: Exact values of the derivatives, their interval enclosures obtained with the approximation of the endpoints of the error term intervals and the widths of intervals (the regular 25-point grid with $\rho_x = \rho_y = 5\mathrm{E}{-6}$)

Deriv.	Interval enclosure of the derivative	Width
$\partial u/\partial x$	[3.21006354171910379E+0000, 3.21006354171960945E+0000] exact ≈ 3.21006354171935371E+0000	5.0565E−13
$\partial u/\partial y$	[3.21006354171910369E+0000, 3.21006354171960958E+0000] exact ≈ 3.21006354171935371E+0000	5.0587E−13
$\partial^2 u/\partial x^2$	[1.05932096499612316E+0001, 1.05932097153302133E+0001] exact ≈ 1.05932096876738672E+0001	6.5368E−08
$\partial^2 u/\partial y^2$	[1.05932096499385176E+0001, 1.05932097153178944E+0001] exact ≈ 1.05932096876738672E+0001	6.5379E−08
$\partial^2 u/\partial x \partial y$	[6.74113340305148239E+0000, 6.74113345648631275E+0000] exact ≈ 6.74113343761064279E+0000	5.3434E−08

Discussion of results and final conclusions

The results obtained with the IGFDs lead to the following general conclusions.

- The interval solution includes the exact value of the derivative in the case of all functions and each numerical experiment. For each example function u and each number of grid points 9, 17, 25, we can find the grid parameter $\rho_x = \rho_y$ such that the widths of interval solutions are the smallest. Their further decrease does not improve the results or even makes them worse.
- The smallest widths of interval solutions are usually obtained with the 9-point grid of points. The larger number of the grid points improves the results when we take $\rho_x = \rho_y$ much smaller than the optimal one.
- The widths of interval solutions are smaller in the case of the regular arrangement of points than the irregular one (in the case of each example function such a difference is equal to about one order of accuracy). Nevertheless, the regular distribution is rarely possible near irregular and complicated boundary. In such a case the interval GFDs are very useful.

Acknowledgments

Thank you very much for your attention ...

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