On an application of an interval finite difference method for solving the heat conduction problem

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Outline

Motivation

Why we use the interval arithmetic and interval methods?

- Inexact initial data can be enclosed in an appropriate interval which endpoints depend on the measurement uncertainties.
- For a real number that cannot be represented exactly in a given floating-point format, we can always find an interval that include such number inside. Furthermore, its left and right endpoints are two neighbouring machine numbers.
- Rounding errors are enclosed in a final interval value, if computations are performed in the floating-point interval arithmetic.
- Finally, for the interval method we assume that the error term of the corresponding conventional method (which is normally neglected) is also included in the final interval solution.

Heat Conduction Problem

We consider the heat conduction problem given by the governing equation

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < L,\ t > 0, \tag{1}$$

subject to the initial condition and the Dirichlet boundary conditions

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{2}$$

$$u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad t > 0. \tag{3}$$

The physical problem concerns the distribution of heat along an isotropic rod of length L (or an isotropic infinite plate of thickness L) over time.

A function $u = u(x,t)$ describes the temperature at a given location x and time t. It is assumed that a temperature within each cross-sectional element of the rod is uniform. Furthermore, the rod is perfectly insulated on its lateral surface.

Heat Conduction Problem

The constant $\alpha = \sqrt{\kappa/(c\rho)}$ is a material-specific quantity and it depends on the heat conductive properties of the material such as:

- the thermal conductivity κ,
- the specific heat c,
- the mass density ρ of the body.

It is assumed that k, c and ρ are independent of the position x in the rod.

Conventional Backward Finite Difference Scheme

Before we adapt the finite difference method for the boundary value problem (1)-(3), we set the maximum time $T_{\max}$, choose two integers n and m and find the mesh constants h and k such as

$$h = L/n, \ \ k = T_{\max}/m.$$

Hence the grid points are (x_i, t_j), where $x_i = ih$ for $i = 0, 1, \ldots, n$ and $t_j = jk$ for $j = 0, 1, \ldots, m$.

Conventional Backward Finite Difference Scheme

Using the Taylor series for each interior grid point we obtain the backward difference formula for $(\partial u/\partial t)(x_i, t_j)$ in the form

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j) - u(x_i, t_{j-1})}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j), \tag{4}$$

and the central difference formula for $\left(\partial^2 u/\partial x^2\right)(x_i, t_j)$ given by

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j), \tag{5}$$

where $\eta_j \in (t_{j-1}, t_j)$ and $\xi_i \in (x_{i-1}, x_{i+1})$.

Conventional Backward Finite Difference Scheme

The heat equation (1) can be expressed at the grid points (x_i, t_j), $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$.

Hence, if we substitute (4)-(5) to the equation (1) expressed at the grid points (x_i, t_j), we obtain

$$\begin{aligned}&(1+2\lambda)\, u(x_i, t_j) - \lambda u(x_{i-1}, t_j) - \lambda u(x_{i+1}, t_j) = u(x_i, t_{j-1}) \\ &\quad -\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) - \alpha^2 \frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j), \\ &i = 1, 2, \ldots, n-1,\ j = 1, 2, \ldots, m,\end{aligned} \tag{6}$$

where $\lambda = \alpha^2 \left(k/h^2\right)$, $\eta_j \in (t_{j-1}, t_j)$, $\xi_i \in (x_{i-1}, x_{i+1})$. Finally, for the initial and boundary conditions (2)-(3) expressed at the grid points (x_i, t_j), we have

$$u(x_i, 0) = f(x_i), \quad i = 0, 1, \ldots, n, \tag{7}$$

$$u(0, t_j) = \varphi_1(t_j), \quad u(L, t_j) = \varphi_2(t_j), \quad j = 1, 2, \ldots, m. \tag{8}$$

Conventional Backward Finite Difference Scheme

For the formulation of the interval couterpart of the conventional backward finite difference method considered, we transform the exact formula (6) with (7)-(8) into the appropriate separate forms according to the position in the grid. We have

$$(1+2\lambda)\,u(x_1,t_j)-\lambda u(x_2,t_j)=\lambda u(x_0,t_j)+u(x_1,t_{j-1})+\widehat{R}_{1,j},\quad i=1,\ j=1,2,\ldots,m, \tag{9}$$

$$(1+2\lambda)\,u(x_i,t_j)-\lambda u(x_{i-1},t_j)-\lambda u(x_{i+1},t_j)=u(x_i,t_{j-1})+\widehat{R}_{i,j},\quad i=2,3,\ldots,n-2,\ j=1,2,\ldots,m, \tag{10}$$

$$(1+2\lambda)\,u(x_{n-1},t_j)-\lambda u(x_{n-2},t_j)=\lambda u(x_n,t_j)+u(x_{n-1},t_{j-1})+\widehat{R}_{n-1,j},\quad i=n-1,\ j=1,2,\ldots,m, \tag{11}$$

where

$$\widehat{R}_{i,j}=-\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_j)-\alpha^2\frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j). \tag{12}$$

Conventional Backward Finite Difference Scheme

The formulas (9)-(11) can be transformed to the matrix representation

$$Cu^{(j)} = u^{(j-1)} + \widehat{E}_C^{(j)} + \widehat{E}_L^{(j)}, \quad j = 1, 2, \ldots, m, \tag{13}$$

where

$$u^{(j)} = \left[u\left(x_1, t_j\right), u\left(x_2, t_j\right), \ldots, u\left(x_{n-1}, t_j\right)\right]^{\mathsf{T}}, \tag{14}$$

$$\widehat{E}_C^{(j)} = [\lambda u(x_0, t_j), 0, \ldots, 0, \lambda u(x_n, t_j)]^{\mathsf{T}}, \quad \widehat{E}_L^{(j)} = \left[\widehat{R}_{1,j}, \widehat{R}_{2,j}, \ldots, \widehat{R}_{n-1,j}\right]^{\mathsf{T}},$$

$$C = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \vdots & 0 & 0 & 0 \\ -\lambda & 1+2\lambda & -\lambda & \vdots & 0 & 0 & 0 \\ 0 & -\lambda & 1+2\lambda & \vdots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \vdots & 1+2\lambda & -\lambda & 0 \\ 0 & 0 & 0 & \vdots & -\lambda & 1+2\lambda & -\lambda \\ 0 & 0 & 0 & \vdots & 0 & -\lambda & 1+2\lambda \end{bmatrix}. \tag{15}$$

Conventional Backward Finite Difference Scheme

$\dim C = (n-1) \times (n-1)$,
$\dim u^{(j)} = \dim \widehat{E}_C^{(j)} = \dim \widehat{E}_L(j) = (n-1) \times 1$,

C – a matrix that is tridiagonal and symmetric; it is also positive definite and strictly diagonally dominant, due to the fact that $\lambda > 0$,

$\widehat{E}_C{}^{(j)}$, $j = 1, 2, \ldots, m$ – the vectors of coefficients depend on the stepsizes h, k, the parameter α and the values of the functions φ_1, φ_2. They are different for each $j = 1, 2, \ldots, m$,

$\widehat{E}_L{}^{(j)}$, $j = 1, 2, \ldots, m$ – the vectors depend on the stepsizes h, k and the values of the appropriate derivatives of u at the midpoints considered; theirs components represent the local truncation error terms of the conventional finite difference method at each mesh point.

Conventional Backward Finite Difference Scheme

Remark 1. Consider the exact formulas (9)-(11) and the corresponding matrix representation (13)-(14). Let $u_{i,j}$ approximate $u(x_i, t_j)$.

If we neglect the error terms $\widehat{R}_{i,j}$, given in the equations (9)-(11) and in the components of the vectors $\widehat{E}_L{}^{(j)}$, then we get the conventional backward finite difference method with the local truncation error $O(h^2 + k)$.

Interval Backward Finite Difference Scheme

For the interval approach we suppose that there exist the intervals $S_{i,j}$, $Q_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$, such that the following relations hold

$$\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) \in S_{i,j}, \quad \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) \in Q_{i,j}. \tag{16}$$

Hence, applying (16) to (12), we have $\widehat{R}_{i,j} \in R_{i,j}$, where

$$R_{i,j} = -\frac{k^2}{2} S_{i,j} - \alpha^2 \frac{kh^2}{12} Q_{i,j}. \tag{17}$$

Interval Backward Finite Difference Scheme

We can formulate the interval backward finite difference method as follows

$$(1+2\lambda)\,U_{1,j} - \lambda U_{2,j} = \lambda U_{0,j} + U_{1,j-1} + R_{1,j}, \qquad (18)$$
$$i = 1,\ j = 1, 2, \ldots, m,$$

$$(1+2\lambda)\,U_{i,j} - \lambda U_{i-1,j} - \lambda U_{i+1,j} = U_{i,j-1} + R_{i,j}, \qquad (19)$$
$$i = 2, 3, \ldots, n-2,\ j = 1, 2, \ldots, m,$$

$$(1+2\lambda)\,U_{n-1,j} - \lambda U_{n-2,j} = \lambda U_{n,j} + U_{n-1,j-1} + R_{n-1,j}, \qquad (20)$$
$$i = n-1,\ j = 1, 2, \ldots, m,$$

where

$$U_{i,0} = F\left(X_i\right), \quad i = 0, 1, \ldots, n, \qquad (21)$$
$$U_{0,j} = \Phi_1\left(T_j\right), \quad U_{n,j} = \Phi_2\left(T_j\right), \quad j = 1, 2, \ldots, m. \qquad (22)$$

X_i, T_j are intervals such that $x_i \in X_i$, $t_j \in T_j$ and $F = F\left(X\right)$, $\Phi_1 = \Phi_1\left(T\right)$, $\Phi_2 = \Phi_2\left(T\right)$ denote interval extensions of $f = f\left(x\right)$, $\varphi_1 = \varphi_1\left(t\right)$, $\varphi_2 = \varphi_2\left(t\right)$.

Interval Backward Finite Difference Scheme

The matrix representation of (18)-(20) is given as follows

$$CU^{(j)} = U^{(j-1)} + E_C^{(j)} + E_L^{(j)}, \quad j = 1, 2, \ldots, m, \tag{23}$$

where

$$\begin{aligned} U^{(j)} &= \left[U_{1,j}, U_{2,j}, \ldots, U_{n-1,j}\right]^{\mathrm{T}}, \\ E_C^{(j)} &= \left[\lambda U_{0,j}, 0, \ldots, 0, \lambda U_{n,j}\right]^{\mathrm{T}}, \quad E_L^{(j)} = \left[R_{1,j}, R_{2,j}, \ldots, R_{n-1,j}\right]^{\mathrm{T}}. \end{aligned} \tag{24}$$

Theorem 1. Let us assume that the local truncation error of the backward finite difference scheme can be bounded by the appropriate intervals at each step. Moreover, let $F = F(X)$, $\Phi_1 = \Phi_1(T)$, $\Phi_2 = \Phi_2(T)$ denote interval extensions of the functions $f = f(x)$, $\varphi_1 = \varphi_1(t)$, $\varphi_2 = \varphi_2(t)$, given in the initial and boundary conditions of the heat conduction problem (1)-(3).

If $u(x_i, 0) \in U_{i,0}$, $i = 0, 1, \ldots, n$, $u(0, t_j) \in \Phi_1(T_j)$, $u(L, t_j) \in \Phi_2(T_j)$, $j = 1, 2, ., m$ and the interval linear system of equations (23) with (24) can be solved with an interval realization of some direct method, then for the interval solutions considered we have

$$u(x_i, t_j) \in U_{i,j}, i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, m.$$

Approximation of the Error Terms

Before we apply the interval method considered, we have to compute the components of the vectors $E_L^{(j)}$. Consequently, the interval values of $S_{i,j}$, $Q_{i,j}$ are required for each mesh point (x_i, t_j).

Determination of the exact values of the endpoints of the error term intervals $S_{i,j}$, $Q_{i,j}$ is still an open problem that deserves further investigation. We propose the method of approximation of the endpoints considered. It is based on the finite difference schemes.

Approximation of the Error Terms

ALGORITHM

Step 1. Take $iter = -1$. Set $\widetilde{n} = n$, $\widetilde{m} = m$ and $\widetilde{h} = h$, $\widetilde{k} = k$. Based on that, the grid points are (x_s, t_q), where $x_s = s\widetilde{h}$, $t_q = q\widetilde{k}$, $s = 0, 1, \ldots, \widetilde{n}$, $q = 0, 1, \ldots, \widetilde{m}$.

Step 2. Take $iter = iter + 1$.

If $iter = 0$, then take $mul = 1$.
Else, if $iter \geq 1$, then take $mul = 2 \cdot mul$.
Then, take $\widetilde{n} = mul \cdot n$, $\widetilde{m} = mul \cdot m$ and $\widetilde{h} = h/mul$, $\widetilde{k} = k/mul$.

Step 3. Solve the initial-boundary value problem with the interval realization of the conventional backward finite difference scheme.
Such interval realization can be understood as the interval backward finite difference method with the components of $E_L^{(q)}$ that represent the local truncation error terms all equal to zero, i.e. we have

$$R_{s,q} = 0.$$

Approximation of the Error Terms

Step 4. Check, if $U_{s,q}^{IA}$ can be further used for the approximation of the endpoints of the error term intervals.

If $iter = 0$, then go back to Step 2.

Else, if $iter \geq 1$, then

(a) Compute the maximum distance $q_{\max}$ between $U_{i,j}^{IA\,(iter-1)}$ and $U_{i,j}^{IA\,(iter)}$. A distance between such two intervals is defined as

$$q_{i,j} = \max\left\{|\underline{U}_{i,j}^{IA\,(iter-1)} - \underline{U}_{i,j}^{IA\,(iter)}|, |\overline{U}_{i,j}^{IA\,(iter-1)} - \overline{U}_{i,j}^{IA\,(iter)}|\right\}. \qquad (25)$$

(b) Choose a tolerance value TOL. If $q_{\max} \leq TOL$, then stop the iteration process and go further to Step 5. Else, go back to Step 2.

EndIf.

Step 5. Compute the endpoints of $S_{i,j}$, $Q_{i,j}$. These intervals are such that for $\eta_j \in (t_{j-1}, t_j)$, $\xi_i \in (x_{i-1}, x_{i+1})$, we have

$$\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) \in S_{i,j} = \left[\underline{S}_{i,j}, \overline{S}_{i,j}\right], \quad \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) \in Q_{i,j} = \left[\underline{Q}_{i,j}, \overline{Q}_{i,j}\right].$$

Approximation of the Error Terms

For an approximation of $\partial^2 u/\partial t^2$ and $\partial^4 u/\partial x^4$ at a given point, we use finite difference schemes. We choose the sixth order ones for the partial derivatives with respect to time t and the fourth order ones for the partial derivatives with respect to space x:

$$\frac{\partial^2 u}{\partial t^2}(x_s, t_q) = \Diamond_{F,\widetilde{k}}\, u(x_s, t_q) + O\left(\widetilde{k}^6\right), \quad \frac{\partial^2 u}{\partial t^2}(x_s, t_q) = \Diamond_{C,\widetilde{k}}\, u(x_s, t_q) + O\left(\widetilde{k}^6\right),$$
$$\frac{\partial^2 u}{\partial t^2}(x_s, t_q) = \Diamond_{B,\widetilde{k}}\, u(x_s, t_q) + O\left(\widetilde{k}^6\right), \tag{26}$$

$$\frac{\partial^4 u}{\partial x^4}(x_s, t_q) = \Diamond_{F,\widetilde{h}}\, u(x_s, t_q) + O\left(\widetilde{h}^4\right), \quad \frac{\partial^4 u}{\partial x^4}(x_s, t_q) = \Diamond_{C,\widetilde{h}}\, u(x_s, t_q) + O\left(\widetilde{h}^4\right),$$
$$\frac{\partial^4 u}{\partial x^4}(x_s, t_q) = \Diamond_{B,\widetilde{h}}\, u(x_s, t_q) + O\left(\widetilde{h}^4\right). \tag{27}$$

We know that $\eta_j \in (t_{j-1}, t_j)$ and $\xi_i \in (x_{i-1}, x_{i+1})$. Hence, we propose to utilize the idea of finite differences for the approximation of the endpoints of the error term intervals, taking the intervals $U_{s,q}^{IA}$ instead of $u(x_s, t_q)$ in (26)-(27).

Approximation of the Error Terms

For a given pair of indexes (i, j), we can always find the indexes (s, q) such that $(x_i, t_j) = (x_s, t_q)$. Hence, we can easily compute the following intervals

$$S^*_{i,j-1} = \Diamond_{\circ,\widetilde{k}}\, U^{IA}_{i,j-1}, \quad S^*_{i,j} = \Diamond_{\circ,\widetilde{k}}\, U^{IA}_{i,j}, \tag{28}$$
$$Q^*_{i-1,j} = \Diamond_{\circ,\widetilde{h}}\, U^{IA}_{i-1,j}, \quad Q^*_{i,j} = \Diamond_{\circ,\widetilde{h}}\, U^{IA}_{i,j}, \quad Q^*_{i+1,j} = \Diamond_{\circ,\widetilde{h}}\, U^{IA}_{i+1,j},$$

where $\circ \in \{F, C, B\}$ and it specifies a forward, central and backward finite difference, respectively.

Finally, we compute the interval hulls of the intervals $S^*_{i,j-1}$, $S^*_{i,j}$ and $Q^*_{i-1,j}$, $Q^*_{i,j}$, $Q^*_{i+1,j}$, respectively. Then, we take the endpoints of the results obtained as the approximations for the endpoints of $S_{i,j}$ and $Q_{i,j}$. We have

$$\underline{S}_{i,j} \approx \min\left\{\underline{S}^*_{i,j-1}, \underline{S}^*_{i,j}\right\}, \quad \overline{S}_{i,j} \approx \max\left\{\overline{S}^*_{i,j-1}, \overline{S}^*_{i,j}\right\}, \tag{29}$$
$$\underline{Q}_{i,j} \approx \min\left\{\underline{Q}^*_{i-1,j}, \underline{Q}^*_{i,j}, \underline{Q}^*_{i+1,j}\right\}, \quad \overline{Q}_{i,j} \approx \max\left\{\overline{Q}^*_{i-1,j}, \overline{Q}^*_{i,j}, \overline{Q}^*_{i+1,j}\right\}.$$

Step 6. Use the error term intervals $S_{i,j}$, $Q_{i,j}$ to compute the components of the interval vectors $E^{(j)}_L$, $j = 1, 2, \ldots, m$.

Numerical experiment

Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{30}$$

subject to the initial distribution and the boundary conditions given as follows:

$$u(x,0) = 1 - 0.8x + \sin \pi x, \quad 0 \le x \le 1, \tag{31}$$

$$u(0,t) = 1, \quad u(1,t) = 0.2, \quad t > 0. \tag{32}$$

The exact solution of (30) with (31)-(32) is of the form

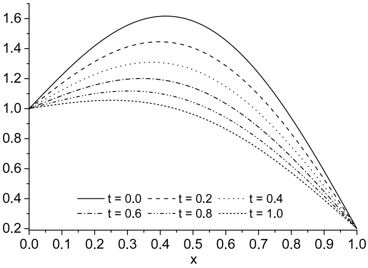
$$u(x,t) = 1 - 0.8x + e^{-t}\sin \pi x. \tag{33}$$

We choose $T_{\max} = 1$.

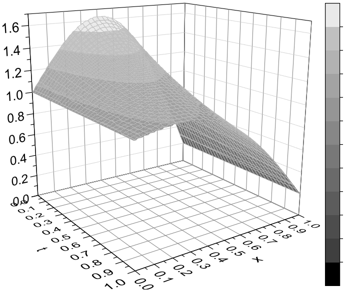
The computations were performed with the C++ libraries (dedicated for the Intel C++ compiler) for the floating-point conversions and interval arithmetic using the double extended precision format.

JANKOWSKA, M.A. (2010) Remarks on Algorithms Implemented in Some C++ Libraries for Floating-Point Conversions and Interval Arithmetic. *Lecture Notes in Computer Science* 6068, 436–445.

Numerical experiment: solution – temperature distribution



(a)



(b)

Figure: Temperature distribution described by the heat conduction problem for: (a) selected values of t; (b) $t \in [0, 1]$.

Numerical experiment: partial derivatives of solution

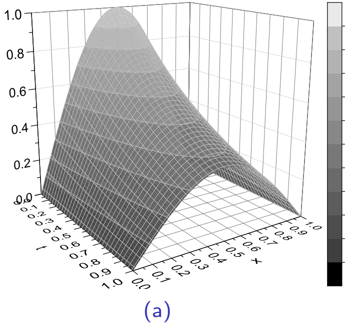


Figure: Partial derivatives of the function u that are present in the error terms: (a) $\partial^2 u/\partial t^2(x, t)$; (b) $\partial^4 u/\partial x^4(x, t)$.

Numerical experiment: assumptions

For a given grid of points (x_i, t_j) defined by the constants n and m, we choose a denser grid of points (x_s, t_q) generated with the constants $\widetilde{n}$ and $\widetilde{m}$. They are used for an approximation of the endpoints of the error term intervals by Algorithm 1.

For most experiments we set $n = m$, $\widetilde{n} = \widetilde{m}$, where $\widetilde{n}$, $\widetilde{m}$ are chosen such that their values are greater or equal to 520 and less or equal to 1000. In this way we have $q_{\max} \leq TOL = 3\text{E-}4$.

Results 1: widths of U at $(x = 0.5, t)$

QUESTION: How a decrease of the step sizes h and k (an increase of n and m) affects the widths of the interval solutions obtained?

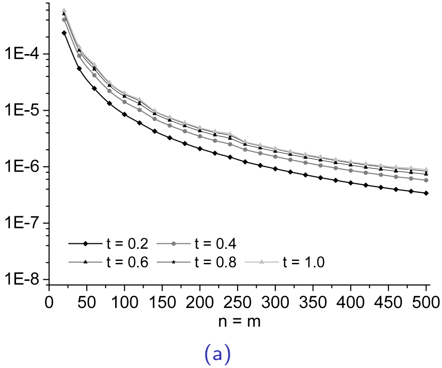


Figure: Widths of the interval solution $U(x = 0.5, t)$ obtained with the interval method considered and different values of n, where: (a) $m = n$; (b) m is such that $k \leq h^2/(2\alpha^2)$.

Results 1, case (a): widths of U at $(x = 0.5, t)$

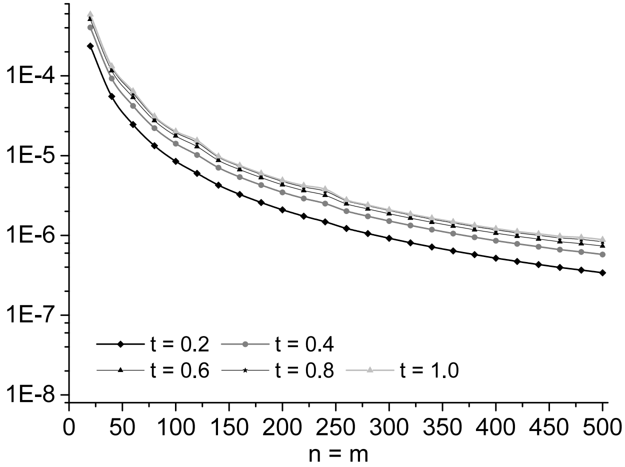


Figure: Case (a): Widths of the interval solution $U(x = 0.5, t)$ obtained with the interval method considered and different values of n, where $m = n$.

Results 1, case (a): widths of U^{IA} and U at $(x, t = 1)$

x	$u(x, t = 1)$	$U^{IA}(x, t = 1)$	width
0.2	1.056E+0	[1.057328494495153E+0,1.057328494495155E+0]	1.98E-15
0.4	1.029E+0	[1.031644890817003E+0,1.031644890817006E+0]	2.78E-15
0.6	8.698E-1	[8.716448908170034E-1,8.716448908170060E-1]	2.57E-15
0.8	5.762E-1	[5.773284944951539E-1,5.773284944951554E-1]	1.56E-15

Table: Values of the exact solution $u(x, t = 1)$ and the interval solution $U^{IA}(x, t = 1)$ obtained with the interval realization of the conventional method, where $h = k = 0.01$.

x	$u(x, t = 1)$	$U(x, t = 1)$	width
0.2	1.0562341E+0	[1.0562272678769E+0,1.0562393576005E+0]	1.20E-05
0.4	1.0298741E+0	[1.0298632472572E+0,1.0298824601789E+0]	1.92E-05
0.6	8.6987413E-1	[8.6986326277521E-1,8.6988244382515E-1]	1.91E-05
0.8	5.7623411E-1	[5.7622729572120E-1,5.7623933001132E-1]	1.20E-05

Table: Values of the exact solution $u(x, t = 1)$ and the interval solution $U(x, t = 1)$ obtained with the interval method, where $h = k = 0.01$.

Results 1, case (b): widths of U at $(x = 0.5, t)$

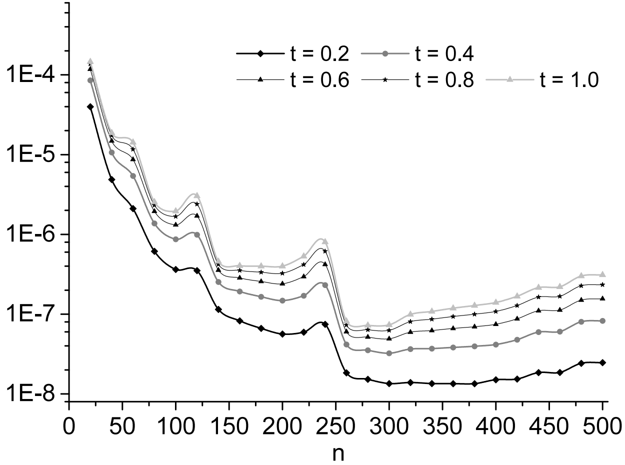


Figure: Case (b): Widths of the interval solution $U(x = 0.5, t)$ obtained with the interval method considered and different values of n, where m is such that $k \leq h^2/(2\alpha^2)$.

Results 1, case (b): widths of U at $(x, t = 1)$

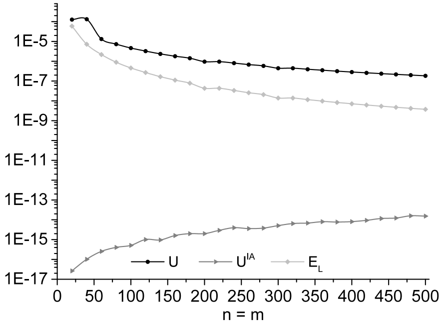
x	$u(x, t = 1)$	$U(x, t = 1)$	*width*
0.2	1.0562341E+0	[1.0562334055759E+0,1.0562348162540E+0]	1.41E-06
0.4	1.0298741E+0	[1.0298731819012E+0,1.0298751092246E+0]	1.92E-06
0.6	8.6987413E-1	[8.6987320002263E-1,8.6987509018268E-1]	1.89E-06
0.8	5.7623411E-1	[5.7623343798116E-1,5.7623478412201E-1]	1.34E-06

Table: Values of the exact solution $u(x, t = 1)$ and the interval solution $U(x, t = 1)$ obtained with the interval method, where $h = 0.01$ ($n = 100$), $k = 4.93\mathrm{E}\text{-}4$.

x	$u(x, t = 1)$	$U(x, t = 1)$	*width*
0.2	1.0562341E+0	[1.0562340838644E+0,1.0562341366531E+0]	5.27E-08
0.4	1.0298741E+0	[1.0298741038421E+0,1.0298741760789E+0]	7.22E-08
0.6	8.6987413E-1	[8.6987410454311E-1,8.6987417537053E-1]	7.08E-08
0.8	5.7623411E-1	[5.7623408508801E-1,5.7623413543092E-1]	5.03E-08

Table: Values of the exact solution $u(x, t = 1)$ and the interval solution $U(x, t = 1)$ obtained with the interval method, where $h = 0.0034$ ($n = 300$), $k = 5.48\mathrm{E}\text{-}5$.

Results 2: widths of U, U^{IA} and E_L at $(x = 0.5, t)$



(a) (b)

Figure: Widths of the interval solutions $U(x = 0.5, t)$, $U^{IA}(x = 0.5, t)$ and the appropriate component $E_L(x = 0.5, t)$ of the vector $E_L(t)$ that contains the error term intervals for: (a) $t = 0.1$; (b) $t = 1$.

Results 3: widths of E_L, S, Q at $(x = 0.5, t)$

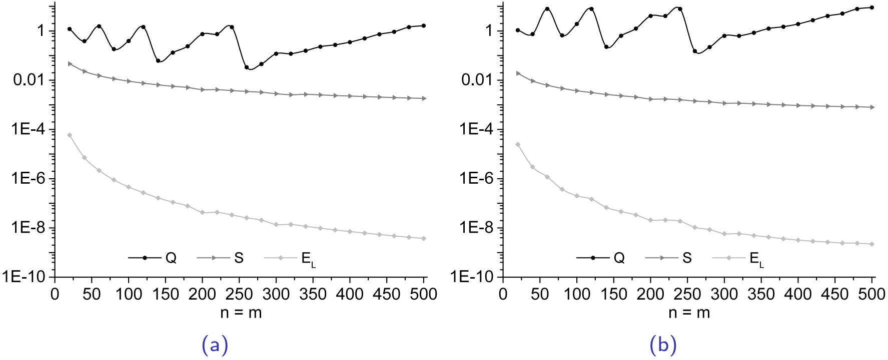


Figure: Widths of the error term interval $E_L(x = 0.5, t)$ and the intervals $S(x = 0.5, t)$, $Q(x = 0.5, t)$ that are all taken into account when the interval solution is produced with the interval method considered, where: (a) $t = 0.1$; (b) $t = 1$.

Summary and conclusions

New ideas proposed in our research included the following points:

- a new algorithm for the approximation of the endpoints of the error term intervals,
- the error term intervals were computed for each point of the domain,
- in the algorithm we applied an iteration process that continues condensing a grid of points as long as a tolerance imposed on some distance between resulting intervals is not met,
- for finding the endpoints of the error term intervals we used interval floating-point solutions obtained from interval realization of conventional backward finite difference method.

Thank you very much for your attention!