Application of an Interval Backward Finite Difference Method for Solving the One-Dimensional Heat Conduction Problem

Malgorzata A. Jankowska[1], Andrzej Marciniak[2] and Tomasz Hoffmann[2]

[1]Poznan University of Technology, Institute of Applied Mechanics
Jana Pawla II 24, 60-965 Poznan, Poland

[2]Poznan University of Technology, Institute of Computing Science
Piotrowo 2, 60-965 Poznan, Poland

Sejmik Młodych Informatyków
18-20 wrzenia, 2013 - Szczecin, Polska

Outline

1. Introduction
2. Heat Conduction Problem
3. Conventional and Interval Backward Finite Difference Methods
4. Results
5. Conclusions

Motivation

Why we use the interval arithmetic and interval methods?

- Inexact initial data can be enclosed in an appropriate interval which endpoints depend on the measurement uncertainties.
- For a real number that cannot be represented exactly in a given floating-point format, we can always find an interval that include such number inside. Furthermore, its left and right endpoints are two neighboring machine numbers.
- Rounding errors are enclosed in a final interval value, if computations are performed in the floating-point interval arithmetic.
- Finally, for the interval method we assume that the error term of the corresponding conventional method (which is normally neglected) is also included in the final interval solution.

One-Dimensional Heat Conduction Problem

Consider the one-dimensional heat conduction problem given by the governing equation

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < L, \ t > 0, \tag{1}$$

subject to the initial condition and the Dirichlet boundary conditions

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{2}$$

$$u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad t > 0. \tag{3}$$

One-Dimensional Heat Conduction Problem

- The heat conduction problem concerns the distribution of heat along an isotropic rod of length L (an isotropic infinite plate of thickness L) over time.
- A function $u = u(x, t)$ describes the temperature at a given location x and time t.
- We assume that a temperature within each cross-sectional element of the rod is uniform.
- Moreover, the rod is perfectly insulated on its lateral surface.
- The constant $\alpha = \sqrt{\kappa}$ is a material-specific quantity. It depends on the thermal diffusivity $\kappa = \lambda/(c\rho)$, where λ is the thermal conductivity, c is the specific heat and ρ is the mass density of the body. It is assumed that λ, c and ρ are independent of the position x in the rod.

Conventional Backward Finite Difference Scheme

Now we establish a grid on the domain. We set the maximum time $T_{\max}$. Then, we choose two integers n, m and we find the mesh constants h, k such as

$$h = L/n,\ k = T_{\max}/m.$$

Hence the grid points are

$$(x_i, t_j),$$

where $x_i = ih$ for $i = 0, 1, \ldots, n$ and $t_j = jk$ for $j = 0, 1, \ldots, m$.

Conventional Backward Finite Difference Scheme

We express the terms of (1) at the grid points (x_i, t_j) and we use the backward finite difference formula for $(\partial u/\partial t)(x_i, t_j)$ and the central finite difference formula for $\left(\partial^2 u/\partial x^2\right)(x_i, t_j)$, together with the appropriate local truncation errors. Hence, if we introduce the notation $\lambda = \alpha^2\left(k/h^2\right)$, we get

$$\begin{aligned}&(1+2\lambda)\,u(x_i,t_j) - \lambda u(x_{i-1},t_j) - \lambda u(x_{i+1},t_j) = u(x_i,t_{j-1})\\&\quad -\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_j) - \alpha^2\frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j),\\&i=1,2,\ldots,n-1,\ j=1,2,\ldots,m,\end{aligned} \tag{4}$$

where $\eta_j \in (t_{j-1}, t_j)$, $\xi_i \in (x_{i-1}, x_{i+1})$ and for the initial and boundary conditions (2)-(3) we have

$$u(x_i,0) = f(x_i), \quad i=0,1,\ldots,n, \tag{5}$$

$$u(0,t_j) = \varphi_1(t_j), \quad u(L,t_j)=\varphi_2(t_j), \quad j=1,2,\ldots,m. \tag{6}$$

Conventional Backward Finite Difference Scheme

For the subsequent formulation of the interval couterpart of the conventional backward finite difference method considered, we transform the exact formula (4) with (5)-(6) into the appropriate separate forms in according to the position in the grid. We have

$$(1+2\lambda)\,u(x_1,t_j)-\lambda u(x_2,t_j)=\lambda u(x_0,t_j)+u(x_1,t_{j-1})+\widehat{R}_{1,j}, \quad i=1,\; j=1,2,\ldots,m, \tag{7}$$

$$(1+2\lambda)\,u(x_i,t_j)-\lambda u(x_{i-1},t_j)-\lambda u(x_{i+1},t_j)=u(x_i,t_{j-1})+\widehat{R}_{i,j}, \quad i=2,3,\ldots,n-2,\; j=1,2,\ldots,m, \tag{8}$$

$$(1+2\lambda)\,u(x_{n-1},t_j)-\lambda u(x_{n-2},t_j)=\lambda u(x_n,t_j)+u(x_{n-1},t_{j-1})+\widehat{R}_{n-1,j}, \quad i=n-1,\; j=1,2,\ldots,m, \tag{9}$$

where

$$\widehat{R}_{i,j}=-\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_j)-\alpha^2\frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j). \tag{10}$$

Conventional Backward Finite Difference Scheme

Note that the formulas (7)-(9) with (10) can be transformed to the following matrix representation

$$Cu^{(j)} = u^{(j-1)} + \widehat{E}_C^{(j)} + \widehat{E}_L^{(j)}, \quad j = 1, 2, \ldots, m, \tag{11}$$

where

$$u^{(j)} = \left[u\left(x_1, t_j\right), u\left(x_2, t_j\right), \ldots, u\left(x_{n-1}, t_j\right)\right]^T, \tag{12}$$

$$\widehat{E}_C^{(j)} = [\lambda u(x_0, t_j), 0, \ldots, 0, \lambda u(x_n, t_j)]^{\mathrm{T}}, \quad \widehat{E}_L^{(j)} = \left[\widehat{R}_{1,j}, \widehat{R}_{2,j}, \ldots, \widehat{R}_{n-1,j}\right]^{\mathrm{T}},$$

The vectors of coefficients $\widehat{E}_C{}^{(j)}$, $j = 1, 2, \ldots, m$, in the formulas (12) depend on the stepsizes h, k, the problem parameter α and the values of the functions φ_1, φ_2. They are different for each $j = 1, 2, \ldots, m$. On the other hand, the vectors $\widehat{E}_L{}^{(j)}$, $j = 1, 2, \ldots, m$, depend on the stepsizes h, k and the values of the appropriate derivatives of u at the midpoints considered. What is most important, the components of $\widehat{E}_L{}^{(j)}$ represent the local truncation error terms of the conventional finite-difference method at each mesh point.

Conventional Backward Finite Difference Scheme

$$C = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \vdots & 0 & 0 & 0 \\ -\lambda & 1+2\lambda & -\lambda & \vdots & 0 & 0 & 0 \\ 0 & -\lambda & 1+2\lambda & \vdots & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 1+2\lambda & -\lambda & 0 \\ 0 & 0 & 0 & \vdots & -\lambda & 1+2\lambda & -\lambda \\ 0 & 0 & 0 & \vdots & 0 & -\lambda & 1+2\lambda \end{bmatrix}. \qquad (13)$$

Note that the matrix C is tridiagonal and symmetric. It is also positive definite and strictly diagonally dominant, due to the fact that $\lambda > 0$.

Interval Backward Finite Difference Scheme

Subsequently, we propose an interval backward finite difference method. It is formulated on the basis of the equations (7)-(9) with (10) or the appropriate matrix representation (11) with (12)-(13). Before that we introduce some assumptions about the values of the derivatives of u at some midpoints considered. Hence, for the interval approach we suppose that there exists the intervals $S_{i,j}$, $Q_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$, such that the following relations hold

$$\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) \in S_{i,j}, \quad \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) \in Q_{i,j}. \tag{14}$$

Hence, applying (14) to (10), we have $\widehat{R}_{i,j} \in R_{i,j}$, where

$$R_{i,j} = -\frac{k^2}{2} S_{i,j} - \alpha^2 \frac{kh^2}{12} Q_{i,j}. \tag{15}$$

Interval Backward Finite Difference Scheme

Then, we can formulate the interval backward finite difference method, related to the equations (7)-(9) with (10), as follows

$$\begin{aligned}&(1+2\lambda)\,U_{1,j}-\lambda U_{2,j}=\lambda U_{0,j}+U_{1,j-1}+R_{1,j},\\ &i=1,\ j=1,2,\ldots,m,\end{aligned} \tag{16}$$

$$\begin{aligned}&(1+2\lambda)\,U_{i,j}-\lambda U_{i-1,j}-\lambda U_{i+1,j}=U_{i,j-1}+R_{i,j},\\ &i=2,3,\ldots,n-2,\ j=1,2,\ldots,m,\end{aligned} \tag{17}$$

$$\begin{aligned}&(1+2\lambda)\,U_{n-1,j}-\lambda U_{n-2,j}=\lambda U_{n,j}+U_{n-1,j-1}+R_{n-1,j},\\ &i=n-1,\ j=1,2,\ldots,m,\end{aligned} \tag{18}$$

where

$$U_{i,0}=F\left(X_i\right),\quad i=0,1,\ldots,n, \tag{19}$$

$$U_{0,j}=\Phi_1\left(T_j\right),\quad U_{n,j}=\Phi_2\left(T_j\right),\quad j=1,2,\ldots,m. \tag{20}$$

Interval Backward Finite Difference Scheme

Similarly, the matrix representation of (16)-(18) with (15) is given as follows

$$CU^{(j)} = U^{(j-1)} + E_C^{(j)} + E_L^{(j)}, \quad j = 1, 2, \ldots, m, \tag{21}$$

where

$$\begin{aligned} U^{(j)} &= \left[U_{1,j}, U_{2,j}, \ldots, U_{n-1,j}\right]^T, \\ E_C^{(j)} &= \left[\lambda U_{0,j}, 0, \ldots, 0, \lambda U_{n,j}\right]^{\mathrm{T}}, \quad E_L^{(j)} = \left[R_{1,j}, R_{2,j}, \ldots, R_{n-1,j}\right]^{\mathrm{T}}. \end{aligned} \tag{22}$$

Interval Backward Finite Difference Scheme

Theorem

Let us assume that the local truncation error of the backward finite difference scheme can be bounded by the appropriate intervals at each step. Moreover, let $F = F(X)$, $\Phi_1 = \Phi_1(T)$, $\Phi_2 = \Phi_2(T)$ *denote interval extensions of the functions* $f = f(x)$, $\varphi_1 = \varphi_1(t)$, $\varphi_2 = \varphi_2(t)$, *given in the initial and boundary conditions of the heat conduction problem (1)-(3). If* $u(x_i, 0) \in U_{i,0}$, $i = 0, 1, \ldots, n$, $u(0, t_j) \in \Phi_1(T_j)$, $u(L, t_j) \in \Phi_2(T_j)$, $j = 1, 2, ., m$ *and the linear system of equations (21) with (22) can be solved with some direct method, then for the interval solutions considered we have* $u(x_i, t_j) \in U_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$.

Interval Backward Finite Difference Scheme

Remark

Taking into consideration the formulas (7)-(9) and (16)-(18) with their appropriate matrix representations (11) and (21), we conclude that the proof of the above theorem is a natural consequence of the thesis of Theorem 2.

Consider a finite system of linear algebraic equations of the form $Ax = b$, where A is an n-by-n matrix, b is an n-dimensional vector and the coefficients of A and b are real or interval values. The existence of the solution to $Ax = b$ is provided by Theorem 2.

Theorem

If we can carry out all the steps of a direct method for solving $Ax = b$ in the interval arithmetic (if no attempted division by an interval containing zero occurs, nor any overflow or underflow), then the system has a unique solution for every real matrix in A and every real matrix in b, and the solution is contained in the resulting interval vector X.

Approximation of the Error Terms

- Note that determination of the exact values of the endpoints of the error term intervals $S_{i,j}$, $Q_{i,j}$ is possible only for some selected examples of the heat conduction problem (1)-(3).
- Generally, for any other case, such issue is still an open problem that deserves further investigation.
- Subsequently, we propose the method of approximation of the endpoints considered. It is based on finite differences that are used to find a minumum and maximum value of the derivatives (present in the error terms) at the points dependent on the intervals that the given midpoints η_j, ξ_i belong to.

Approximation of the Error Terms

We assumed that the relations (14) hold for the appropriate intervals $S_{i,j}$, $Q_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$. They are such that for $\eta_j \in (t_{j-1}, t_j)$, $\xi_i \in (x_{i-1}, x_{i+1})$, we have

$$\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) \in S_{i,j} = \left[\underline{S}_{i,j}, \overline{S}_{i,j}\right], \quad \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) \in Q_{i,j} = \left[\underline{Q}_{i,j}, \overline{Q}_{i,j}\right].$$

We can choose the endpoints $\underline{S}_{i,j}$ and $\overline{S}_{i,j}$ as

$$\underline{S}_{i,j} \approx \min\left(S^*_{i,j-1}, S^*_{i,j-1/2}, S^*_{i,j}\right), \quad \overline{S}_{i,j} \approx \max\left(S^*_{i,j-1}, S^*_{i,j-1/2}, S^*_{i,j}\right), \tag{23}$$

and then, in the similar way, the endpoints $\underline{Q}_{i,j}$ and $\overline{Q}_{i,j}$

$$\underline{Q}_{i,j} \approx \min\left(Q^*_{i-1,j}, Q^*_{i,j}, Q^*_{i+1,j}\right), \quad \overline{Q}_{i,j} \approx \max\left(Q^*_{i-1,j}, Q^*_{i,j}, Q^*_{i+1,j}\right), \tag{24}$$

where

$$S^*_{i,j} = \frac{\partial^2 u}{\partial t^2}(x_i, t_j), \quad Q^*_{i,j} = \frac{\partial^4 u}{\partial x^4}(x_i, t_j). \tag{25}$$

Numerical experiment

Consider the heat conduction problem given by the governing equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \; t > 0, \tag{26}$$

and the initial and boundary conditions

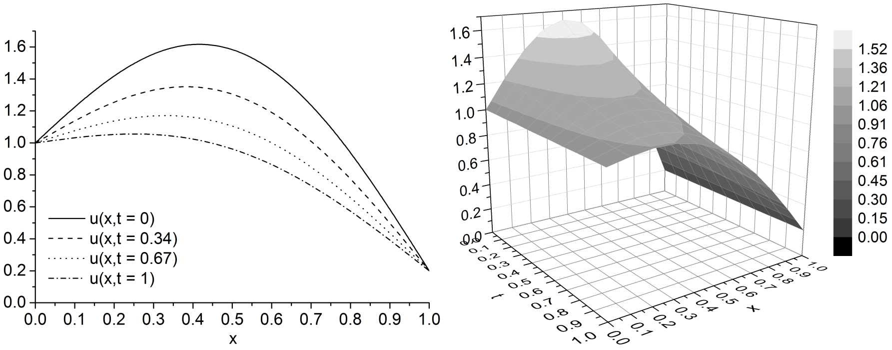
$$u(x,0) = 1 - 0.8x + \sin \pi x, \quad 0 \le x \le 1, \tag{27}$$

$$u(0,t) = 1, \quad u(1,t) = 0.2, \quad t > 0. \tag{28}$$

The analytical solution of the problem is of the following form

$$u(x,t) = 1 - 0.8x + e^{-t}\sin \pi x. \tag{29}$$

Numerical experiment



(a) (b)

Figure: Temperature distribution described by the heat conduction problem for: (a) selected values of t; (b) $t \in [0, 1]$

Numerical experiment

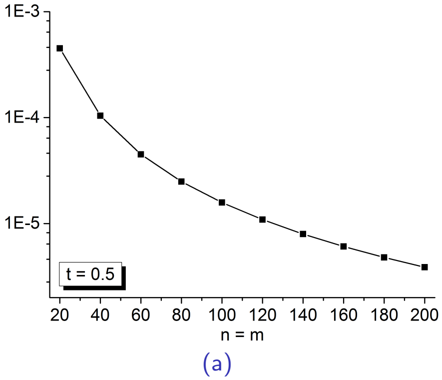
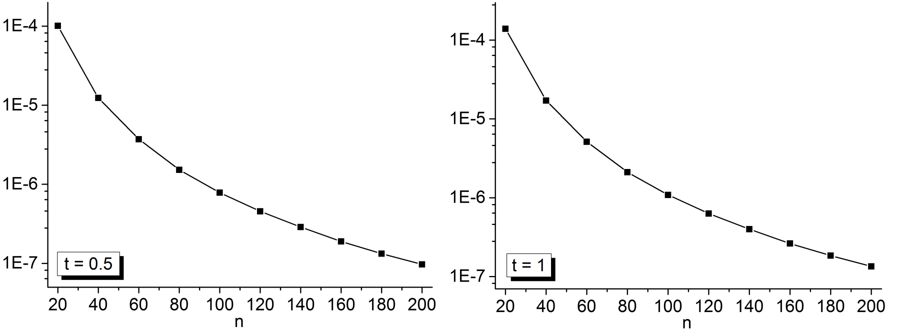


Figure: Widths of the interval solutions: (a) $U(x, t = 0.5)$, (b) $U(x, t = 1)$ obtained with the interval backward finite difference method for the heat conduction problem and different values of $n = m$.

Numerical experiment



(a) (b)

Figure: Widths of the interval solutions: (a) $U(x, t = 0.5)$, (b) $U(x, t = 1)$ obtained with the interval backward finite difference method for the heat conduction problem and different values of n, where m are such as $k \leq h^2/(2\alpha^2)$.

Numerical experiment

x	$u(x, t=1)$	$U^C(x, t=1)$	width
0.1	1.033680E+0	[1.034256351106E+0, 1.034256351106E+0]	1.15E-15
0.2	1.056234E+0	[1.057328494495E+0, 1.057328494495E+0]	1.98E-15
0.3	1.057620E+0	[1.059127010626E+0, 1.059127010626E+0]	2.52E-15
0.4	1.029874E+0	[1.031644890817E+0, 1.031644890817E+0]	2.79E-15
0.5	9.678794E-1	[9.697413190404E-1, 9.697413190404E-1]	2.79E-15
0.6	8.698741E-1	[8.716448908170E-1, 8.716448908170E-1]	2.57E-15
0.7	7.376207E-1	[7.391270106263E-1, 7.391270106263E-1]	2.15E-15
0.8	5.762341E-1	[5.773284944951E-1, 5.773284944951E-1]	1.56E-15
0.9	3.936809E-1	[3.942563511061E-1, 3.942563511061E-1]	8.31E-16

Table: Values of the exact solution and the interval solution $U^C(x, t=1)$ obtained with the interval realization of the conventional backward finite difference method for the heat conduction problem, where $h = 1\text{E-}2$ and $k = 1\text{E-}2$.

Numerical experiment

x	$u(x, t=1)$	$U(x, t=1)$	*width*
0.1	1.033680E+0	[1.033679103041E+0, 1.033685254214E+0]	6.15E-6
0.2	1.056234E+0	[1.056230573203E+0, 1.056242133869E+0]	1.15E-5
0.3	1.057620E+0	[1.057615928504E+0, 1.057631686586E+0]	1.57E-5
0.4	1.029874E+0	[1.029868569829E+0, 1.029886969356E+0]	1.83E-5
0.5	9.678794E-1	[9.678736104334E-1, 9.678929051603E-1]	1.92E-5
0.6	8.698741E-1	[8.698685698291E-1, 8.698869693562E-1]	1.83E-5
0.7	7.376207E-1	[7.376159285043E-1, 7.376316865860E-1]	1.57E-5
0.8	5.762341E-1	[5.762305732034E-1, 5.762421338693E-1]	1.15E-5
0.9	3.936809E-1	[3.936791030410E-1, 3.936852542143E-1]	6.15E-6

Table: Values of the exact solution and the interval solution $U(x, t=1)$ obtained with the interval backward finite difference method for the heat conduction problem, where $h = 1\text{E-}2$ and $k = 1\text{E-}2$.

Conclusions

The main features of the conventional backward finite difference method are given as follows:

- the method is based on some finite difference representation of the derivatives given in the governing equation and the initial and boundary conditions,
- the local truncation error of the method is known for each step of the method, but it is neglected in the conventional approach,
- the finite difference scheme obtained, is of second order with space and first in time, i.e. $O(h^2 + k)$,
- since the error term is neglected, the method produces only approximations of $u(x_i, t_j)$.

Conclusions

The main features of the interval backward finite difference method are given as follows:

- the interval method is based on the conventional scheme,
- it is developed in terms of interval arithmetic and interval analysis,
- the local truncation error of the conventional method is bounded by some error term intervals, hence we can prove that the exact solution of the problem belongs to the appropriate interval solutions obtained,
- all interval methods have to be implemented in the floating-point interval arithmetic.

Thank you very much for your attention!