An Interval Version of the Crank-Nicolson Method – the First Approach

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Abstract To study the heat or diffusion equation it is often used the Crank-Nicolson method which is unconditionally stable and has order of convergence $O(k^2 + h^2)$, where k and h are mesh constants. Unfortunately, using this method in conventional floating-point arithmetic we get solutions including not only the method error, but also representation and rounding error, Therefore, we propose an interval version of Crank-Nicolson method from which we would like to obtain solutions including the method error. Applying such a method in interval floating-point arithmetic one can get solutions including all possible numerical errors. A numerical example is presented.

Keywords heat equation, Crank-Nicolson method, interval methods, floating-point interval arithmetic

1 Introduction

In a number of our previous paper we developed interval methods for solving the initial value problem (see e.g. [1] – [8]). These methods have been based on conventional Runge-Kutta and multistep methods. We have summarized our previous research in [10].

Now, our affords are directed to construct similar methods for solving a variety of problems in partial-differential equations. In [9] we have proposed an interval method for solving the Poisson equation. Here we present a proposition of interval method based on the Crank-Nicolson scheme for solving the heat equation.

2 The Heat Equation and Crank-Nicolson Method

The parabolic partial-differential equation we will consider is the heat or diffusion equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad a \le x \le b, \quad t > 0, \tag{1}$$

subject to the conditions

$$u(a,t) = 0, \quad u(b,t) = 0, \quad t > 0,$$

and

$$u(x,0) = f(t), \quad a \le x \le b.$$

The approach one uses to approximate the solution to this problem involves finite differences.

First we select two mesh constant h and k, with the stipulation that $m = (b - a)/h$ is an integer. The grid points are (x_i, t_j), where $x_i = ih$ for $i = 0, 1, \ldots, m$, and $t_j = jk$ for $j = 0, 1, 2, \ldots$.

Using the forward-difference method at the jth step in t, we get

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j),$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j),$$

where

$$\mu_j \in (t_j, t_{j+1}), \quad \xi_i \in (x_{i-1}, x_{i+1}),$$

and taking the backward-difference method at the $(j + 1)$st step in t, we obtain

$$\frac{\partial u}{\partial t}(x_i, t_{j+1}) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_j),$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_{j+1}) = \frac{u(x_i + h, t_{j+1}) - 2u(x_i, t_{j+1}) + u(x_i - h, t_{j+1})}{h^2} -$$

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$$-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\tilde{\xi}_i, t_{j+1}),$$

where

$$\tilde{\mu}_j \in (t_j, t_{j+1}), \quad \tilde{\xi}_i \in (x_{i-1}, x_{i+1}).$$

Averaging the above formulas and omitting the local truncation errors, we get the following method:

$$\begin{aligned} v_{i,j+1} - v_{ij} - \frac{\alpha^2 k}{2h^2}(v_{i+1,j} - 2v_{ij} + v_{i-1,j} \\ + v_{i+1,j+1} - 2v_{i,j+1} + v_{i-1,j+1}) = 0, \end{aligned} \tag{2}$$

where v_{ij} approximates $u(x_i, y_j)$. This method is known as the Crank-Nicolson method and has local truncation error of order $O(k^2 + h^2)$, provided that the usual differentiability conditions are satisfied.

3 An Interval Crank-Nicolson Method

Taking the local truncation errors into consideration, the equation (2) can be written in the form

$$\begin{aligned} &-\frac{\lambda}{2}v_{i-1,j+1} + (1+\lambda)v_{i,j+1} - \frac{\lambda}{2}v_{i+1,j+1} \\ &= \frac{\lambda}{2}v_{i-1,j} + (1-\lambda)v_{ij} + \frac{\lambda}{2}v_{i+1,j} \\ &\quad + \frac{k^2}{4}\left[\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j) - \frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_j)\right] \\ &\quad - \frac{\lambda h^4}{24}\left[\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) + \frac{\partial^4 u}{\partial x^4}(\tilde{\xi}_i, t_{j+1})\right], \end{aligned} \tag{3}$$

where $\lambda = \alpha^2 k / h^2$.

Let us assume that

$$\frac{\partial^3 u}{\partial x^2 \partial t} = \frac{\partial^3 u}{\partial t \partial x^2},$$

and

$$\left|\frac{\partial^3 u}{\partial t \partial x^2}\right| \le M, \tag{4}$$

where M = const. From (1) we have

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2\frac{\partial^3 u}{\partial t \partial x^2}(x,t) = 0,$$

$$\frac{\partial^3 u}{\partial x^2 \partial t}(x,t) - \alpha^2\frac{\partial^4 u}{\partial x^4}(x,t) = 0.$$

Thus, from (4) it follows that

$$\left|\frac{\partial^2 u}{\partial t^2}(x,t)\right| = \alpha^2\left|\frac{\partial^3 u}{\partial t \partial x^2}(x,t)\right| \le \alpha^2 M,$$

$$\left|\frac{\partial^4 u}{\partial x^4}(x,t)\right| = \frac{1}{\alpha^2}\left|\frac{\partial^3 u}{\partial x^2 \partial t}(x,t)\right| \le \frac{1}{\alpha^2}M,$$

and it means that

$$\frac{\partial^2 u}{\partial t^2}(x,t) \in \alpha^2[-M, M],$$

and

$$\frac{\partial^4 u}{\partial x^4}(x,t) \in \frac{1}{\alpha^2}[-M, M].$$

Taking into account the above relations, we define an interval version of the Crank-Nicolson methods as follows:

$$\begin{aligned} &-\frac{\lambda}{2}V_{i-1,j+1} + (1+\lambda)V_{i,j+1} - \frac{\lambda}{2}V_{i+1,j+1} \\ &= \frac{\lambda}{2}V_{i-1,j} + (1-\lambda)V_{ij} + \frac{\lambda}{2}V_{i+1,j} \\ &\quad + \frac{k}{2}\left(\alpha^2 k - \frac{h^2}{6}\right)[-M, M], \end{aligned} \tag{5}$$

where $V_{ij} = [\underline{v}_{ij}, \overline{v}_{ij}]$.

The system of equations (5) is linear with a positive definite, symmetric, strictly diagonally dominant and tridiagonal matrix. It can be solved by an interval version of Crout reduction method.

In practice it can be difficult to determine the constant M since $u(x, t)$ is unknown. If it is impossible to determine M from any physical or other conditions of the problem considered, we propose to solve the problem by the conventional Crank-Nicolson method (2) and take

$$\begin{aligned} M \approx \frac{15}{kh^2} \max_{\substack{i=1,2,\ldots,m-1 \\ j=1,2,\ldots,n-1}} \bigl| v_{i+1,j} - v_{i+1,j-1} \\ - 2(v_{ij} - v_{i,j-1}) + v_{i-1,j} - v_{i-1,j-1} \bigr|. \end{aligned}$$

4 A Numerical Example

To have a view on interval solutions obtained, let us consider a problem for which the exact solution is known. Let the method (5) be used to approximate the solution to the problem consisting of the equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0,$$

subject to the conditions

$$u(x,0) = \cos\frac{\pi x}{2}, \quad -1 \le x \le 1,$$

and

$$u(-1,t) = u(1,t) = 0, \quad t \ge 0.$$

The exact solution of the above problem is as follows:

$$u(x,t) = \exp\left(-\frac{\pi^2 t}{4}\right)\cos\left(\frac{\pi x}{2}\right). \quad (6)$$

The graph of this solution for $0 \le t \le 0.1$ is presented in Figure 1, and some particular values are following:

$$u(0, 0.05) \approx 0.88393649689751144,$$
$$u(0, 0.1) \approx 0.78134373054744425.$$

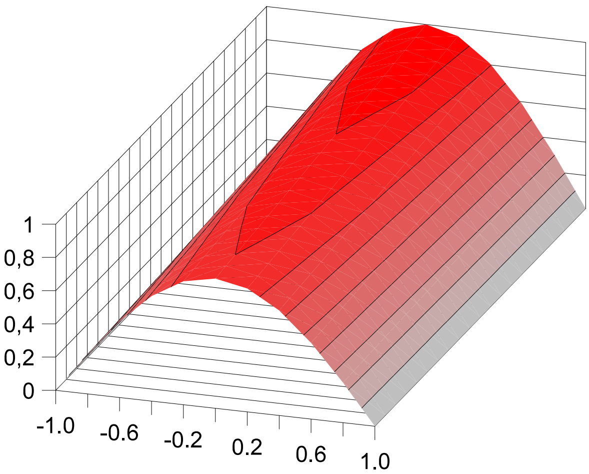

Figure 1: The graph of the function (6)

Using the method (5) with $M = \pi^4/16$, $m = 20$, i.e. $h = 0.1$, and $k = 0.05$, and carried out all calculations in floating-point interval arithmetic (using the Delhi Pascal unit *IntervalArithmetic* described in [10]) we obtain

$$V(0, 0.05) = [0.88086446763895959, 0.88718106618855630].$$

The width of this interval is approximately equal to 6.3×10^{-3}. For the same h and $k = 0.005$ we get

$$V(0, 0.05) = [0.88390572217259396, 0.88441266516723841],$$
$$V(0, 0.1) = [0.78123648428033894, 0.78223847522173376].$$

The widths of these intervals are approximately equal to 5.1×10^{-4} and 6.3×10^{-3} respectively. Unfortunately, for larger values of t we observe a sudden increase of the widths of interval solutions.

Let us note that the exact solution belongs to the interval solutions obtained. Although in many other numerical experiments carried out we have observed the same, it is not true in general. We have a number of examples in which the exact solution is outside interval solutions V_{ij} obtained by the method (5). It follows from the fact that it is impossible to prove that $u(x_i, t_j) \in V_{ij}$.

5 Conclusions and Further Studies

The interval method (5) based on the conventional Crank-Nicolson scheme is only a proposition for solving parabolic partial-differential equations such as the heat equation. Applying this method in floating-point interval arithmetic we can automatically include into interval solutions the representation and rounding errors.

Since for the method (5) we are not capable of proving that the exact solution belongs to the interval solutions obtained, the presented method should be modified to fulfil this necessary condition. Moreover, the method should be also modified with respect to the increase of interval widths for the larger number of steps.

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