On the application of directed interval arithmetic for solving elliptic BVP

The 28th International Conference on Applications of Computer Algebra

Tomasz Hoffmann [1] Andrzej Marciniak [2,3]
Małgorzata A. Jankowska [4]

[1]Poznan Supercomputing and Networking Center, Poznan, Poland
[2]Department of Computer Science, The President Stanisław Wojciechowski Calisia University, Kalisz, Poland
[3]Institute of Computing Science, Poznan University of Technology, Poznan, Poland
[4]Institute of Applied Mechanics, Poznan University of Technology, Poznan, Poland

July 19 2023

Significance of the problem

The elliptic partial differential equations (PDEs), particularly the Poisson equation and its generalized form, describe many problems in such fields as:

- fluid mechanics,
- electrostatics,
- image processing (e.g., the composition of two images).

The usage of interval arithmetic allows us to obtain complete information about numerical errors. In practice, this gives us automatically estimate of the accuracy of solutions by the width of the intervals.

Research problem and purpose of the work

Problem

Development of methods for solving elliptic partial differential equations (using Poisson's equation as an example) in interval arithmetic.

Goal

The results obtained should include information about all numerical errors, that is, they should include the following:

- input errors (e.g., measurement inaccuracies),
- representation errors,
- method errors (e.g. estimated with constants),
- rounding errors (arising in the process of calculations).

Problem - elliptic PDEs of the second order

Poisson Equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y), \quad \alpha_1 < x < \alpha_2, \beta_1 < y < \beta_2, \tag{1}$$

with Dirichlet boundary conditions

$$u|_\Gamma = \varphi(x,y) = \begin{cases} \varphi_1(y), & x = \alpha_1, \\ \varphi_2(x), & y = \beta_1, \\ \varphi_3(y), & x = \alpha_2, \\ \varphi_4(x), & y = \beta_2. \end{cases}$$

Problem - elliptic PDEs of the second order

Generalized Poisson Equation

$$a(x,y)\frac{\partial^2 u}{\partial x^2}(x,y)+b(x,y)\frac{\partial^2 u}{\partial y^2}(x,y)=f(x,y), \quad 0<x<\alpha, 0<y<\beta, \tag{2}$$

with Dirichlet boundary conditions (as in equation (1)) and a condition

$$a(x,y)\cdot b(x,y)>0.$$

Also, in both equations ((1) i (2)) there are equalities:

$$\varphi_1(0)=\varphi_2(0), \varphi_2(\alpha)=\varphi_3(0), \varphi_3(\beta)=\varphi_4(\alpha), \varphi_4(0)=\varphi_1(\beta).$$

Problem - elliptic PDEs of the second order

Nakao's Equations

The class of elliptic equations, for which the method (belonging to the FEM class) was presented by M. T. Nakao [11].
These equations can be generally written in the form

$$u + \mathbf{b}\nabla u + cu = -f, \tag{3}$$

where $\mathbf{b} = [b_i(x, y)]$ is a vector of functions being coefficients of the equation.

Research hypothesis

Main research hypothesis

The use of interval arithmetic for solving the Poisson equation (1) and its generalization (2) makes it possible to automatically take into account various numerical errors inside the obtained interval solutions. (H1)

Research hypothesis

Auxiliary hypotheses

Automatic solution estimation methods developed as part of hypothesis verification (H1) can be generalized for the case of second order elliptic partial differential equations (PDEs). (H2)

The use of directed interval arithmetic in calculations makes it possible to obtain better (more accurate) estimates than in the proper one. (H3)

State of the art – existing methods

For the Poisson equation, there are many methods to solve it in the floating point arithmetic. In general, they belong to two main types:

- finite difference methods (FDM),
- finite element methods (FEM).

There are well-known applications of interval arithmetic, including for ordinary and partial differential equations (see: [7, 9, 13]), but they differ from the proposed approach - no attempt has yet been made to account for possible numerical errors holistically and perform all computations using intervals.

State of the art – interval arithmetic

The proper interval arithmetic

The standardized (IEEE-1788) interval arithmetic (described by Moore [8] and Kearfott [3]), where an element is represented as a pair - the lower and upper ends of an interval.

The directed interval arithmetic

The interval arithmetic described in the works of E. Popova [14] and S. Markov [6]. From the theoretical point of view, the most important is that we generalize here the notion of an interval, allowing it to include also the situations where the left end is greater than the right end.

State of the art – Nakao's method overview

Algorithm 1 Nakao's method for verification and estimation of elliptic PDE solutions (for details see: [4])

1: $u_h^{(0)} := GalerkinApprox(f, \phi, n)$
2: $k := 1$
3: $\{EndCondition(u_h^{(k)}, u_h^{(k-1)})$ function checks stop condtions$\}$
4: **while not** $EndCondition(u_h^{(k)}, u_h^{(k-1)})$ **do**
5: $\left(\nabla u_h^{(k)}, \nabla \phi_{ij}\right) := \left(cu_h^{(k-1)} + f, \phi_{ij}\right) + [-1,1]\, h\alpha^{(k-1)} \left\|\phi_{ij}\right\|_{L^2(0,1)\times(0,1)}$
6: $\alpha^{(k)} := h\left(\left\|cu_h^{(k-1)} + f\right\|_{L^2(0,1)\times(0,1)} + h\left\|c\right\|_{L^\infty(0,1)\times(0,1)}\alpha^{(k-1)}\right)$
7: $k := k + 1$
8: **end while**
9: $\{$function $DeltaExtension(u_h^{(k)})$ implements formula for estimate method errors$\}$
10: $u_h^{(k)} := DeltaExtension(u_h^{(k)})$
11: **return** $u_h^{(k)}$

State of the art – Nakao's method overview

We have to note that the presented Algorithm 1 shows that the method proposed by M. T. Nakao in the articles series [10, 11, 12] is involved:

- complex mathematical apparatus including functional analysis (like L^∞, L^2 norms)
- many restrictions for the function f (has to be sufficiently smooth)
- initial solution is based on the Galerkin approximation
- computations are performed in the floating-point arithmetic.

The concept of solving the problem

The proposed approach to solving the problem includes:

- implementations of two types of interval arithmetic – proper and directed (step 0)
- taking the existing finite difference methods (second- and fourth-order methods) for floating-point arithmetic as a starting point (step 1)
- preparation of interval methods with the estimation of the method error (step 2) and its inclusion in the calculations (step 3)

We have to carry out the entire calculations in the interval arithmetic. As the results obtained in the form of ranges should contain full information about numerical errors (which is the solution to the problem).

Interval methods construction – step 1

Central difference method (for generalized Poisson equation)

$$a(x_i, y_j) \cdot \left[\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) \right] +$$
$$b(x_i, y_j) \cdot \left[\frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} - \frac{k^2}{12} \frac{\partial^4 u}{\partial y^4}(x_i, \eta_j) \right]$$
$$= f(x_i, y_j), \tag{4}$$

where $\xi_i \in (x_{i-1}, x_{i+1}), \eta_j \in (y_{j-1}, y_{j+1})$.

Interval methods construction – step 1

Central difference method - boundary conditions

The boundary conditions are given as follows:

$$
\begin{aligned}
u(\alpha_1, y_j) &= \varphi_1(y_j),\ \text{dla } j = 0, 1 \cdots, m,\\
u(x_i, \beta_1) &= \varphi_2(x_i),\ \text{dla } i = 1, 2 \cdots, n-1,\\
u(\alpha_2, y_j) &= \varphi_3(y_j),\ \text{dla } j = 0, 1 \cdots, m,\\
u(x_i, \beta_2) &= \varphi_4(x_i),\ \text{dla } i = 1, 2 \cdots, n-1.
\end{aligned} \tag{5}
$$

Interval methods construction – step 1

Central difference method in floating-point arithmetic

The equation (4) can be written in a simplified form, where the error components of the method are omitted:

$$a(x_i, y_j) \cdot \left[\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} \right] + \\ + b(x_i, y_j) \cdot \left[\frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} \right] = \\ = f(x_i, y_j). \tag{6}$$

Interval methods construction – step 2

An interval version of the central difference method (for the generalized Poisson equation)

In the interval arithmetic, we also consider the error of the method and then the equation (4) takes the form

$$a(x_i, y_j) \cdot \left[\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} \right] +$$
$$+ b(x_i, y_j) \cdot \left[\frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} \right] =$$
$$= f(x_i, y_j) + a(x_i, y_j) \cdot \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + b(x_i, y_j) \cdot \frac{k^2}{12} \frac{\partial^4 u}{\partial y^4}(x_i, \eta_j). \tag{7}$$

Interval methods construction – step 2

Estimation of the error of methods

We take the estimation of the method error with the following constants:

$$\left|\frac{\partial^4 u}{\partial x^4}(x,y)\right| \leq M \text{ for all } \alpha_1 \leq x \leq \alpha_2 \ \wedge \ \beta_1 \leq y \leq \beta_2,$$
$$\left|\frac{\partial^4 u}{\partial y^4}(x,y)\right| \leq N \text{ for all } \alpha_1 \leq x \leq \alpha_2 \ \wedge \ \beta_1 \leq y \leq \beta_2. \tag{8}$$

Interval methods construction – step 2

Estimation of the error of methods

We propose to determine the constants M and N experimentally, using the following relationships:

$$M_h = max_{i,j}\left\{\frac{6u_{i,j} - 4u_{i-1,j} - 4u_{i+1,j} + u_{i-2,j} + u_{i+2,j}}{h^4}\right\},$$
$$N_k = max_{i,j}\left\{\frac{6u_{i,j} - 4u_{i,j-1} - 4u_{i,j+1} + u_{i,j-2} + u_{i,j+2}}{k^4}\right\}, \quad (9)$$

then $M = \lim_{h\to 0} M_h, \; N = \lim_{k\to 0} N_k$.

Interval methods construction – step 3

Interval version of the method - proper arithmetic

$$k^2 A_{i,j} U_{i+1,j} + h^2 B_{i,j} U_{i,j+1} - 2(k^2 A_{i,j} + h^2 B_{i,j}) U_{i,j} + k^2 A_{i,j} U_{i-1,j} + \\ + h^2 B_{i,j} U_{i,j-1} = h^2 k^2 \left\{ F_{i,j} + \frac{h^2 A_{i,j}}{12} [-M, M] + \frac{k^2 B_{i,j}}{12} [-N, N] \right\}. \qquad (10)$$

Interval version of the method - directed arithmetic

$$k^2 A_{i,j} U_{i+1,j} + h^2 B_{i,j} U_{i,j+1} - 2(k^2 A_{i,j} + h^2 B_{i,j}) U_{i,j} + k^2 A_{i,j} U_{i-1,j} + \\ + h^2 B_{i,j} U_{i,j-1} = h^2 k^2 \left\{ F_{i,j} + \frac{h^2 A_{i,j}}{12} [M, -M] + \frac{k^2 B_{i,j}}{12} [N, -N] \right\}. \qquad (11)$$

Interval methods construction – remarks

The presented methods (10) and (11), for the Poisson and generalized Poisson equation, are described in detail in our works [1] and [2], respectively.

For the Nakao equation (3) we proposed an interval method [5] where the following constants where used as errors estimates:

$$\left|\frac{\partial^3 u}{\partial x^2 \partial y}(x,y)\right| \leq P, \quad \left|\frac{\partial^3 u}{\partial x \partial y^2}(x,y)\right| \leq Q, \quad \left|\frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y)\right| \leq R.$$

Example 1

Consider the generalized Poisson equation (2) defined in the area $1 \leq x \leq 2, \;\; 1 \leq y \leq 2$ and assume

$$\begin{aligned} f(x,y) &= x^2y^2(3y^2 + 2x^2y^2 - 3x^2), \\ a(x,y) &= xy^3 e^{\frac{x^2+y^2}{2}}, \\ b(x,y) &= x^3 y e^{\frac{x^2-y^2}{2}}, \end{aligned} \tag{12}$$

with boundary conditions

$$\begin{aligned} \varphi_1(y) &= ye^{\frac{1-y^2}{2}} \;\; \varphi_2(x) = xe^{\frac{x^2-1}{2}}, \\ \varphi_3(y) &= 2ye^{\frac{4-y^2}{2}} \;\; \varphi_4(x) = 2xe^{\frac{x^2-4}{2}}, \end{aligned} \tag{13}$$

then the analytical solution is expressed by the formula

$$u(x,y) = xye^{\frac{x^2-y^2}{2}}. \tag{14}$$

Example 1 – numerical results

Table 1. The results in proper interval arithmetic at $x = y = 1.5$. The exact solution is $u(1.5, 1.5) = 2.3714825526419476\text{e-}01$

$m = n$	$U(0.5, 0.5)$	Width
20 (PIA)	[0.23706842889509303, 0.23720354321116375]	1.3511431607070813e-04
20 (DIA)	[0.23706842889509324, 0.23720354321116354]	1.3511431607028675e-04
40 (PIA)	[0.23712827567900367, 0.23716210346652419]	3.3827787520512100e-05
40 (DIA)	[0.23712827567900447, 0.23716210346652339]	3.3827787518916087e-05
60 (PIA)	[0.23713937354963539, 0.23715441218408559]	1.5038634450192429e-05
60 (DIA)	[0.23713937354963727, 0.23715441218408377]	1.5038634446434287e-05
80 (PIA)	[0.23714325892563301, 0.23715171895776956]	8.4600321365399674e-06
80 (DIA)	[0.23714325892563648, 0.23715171895776609]	8.4600321295948394e-06
100 (PIA)	[0.23714505749665764, 0.23715047215434425]	5.4146576865966704e-06
100 (DIA)	[0.23714505749666336, 0.23715047215433854]	5.4146576751736519e-06

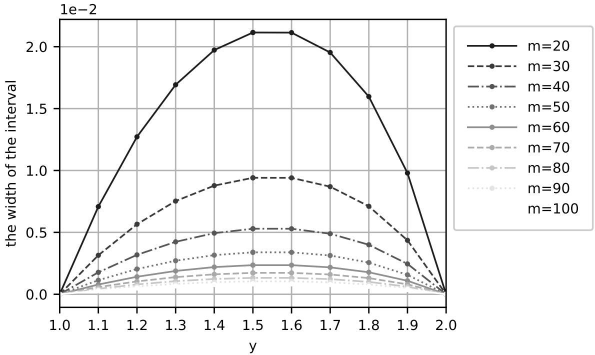

Figure 1. The widths of the resulting intervals obtained by proper interval arithmetic at the point $x = 1,5$

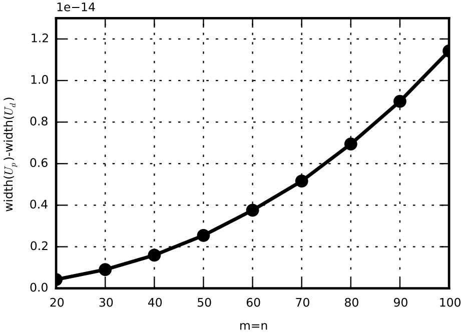

Figure 2. Differences in widths of resultant intervals obtained in directed and proper interval arithmetic for a point $(x, y) = (1,5; 1,5)$

Example 2

Consider an elliptic equation defined over the area of the $0 \leq x \leq 1,\ \ 0 \leq y \leq 1$ expressed by the formula

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) + \frac{5}{4}\pi^2 u(x,y) = -\pi \sin(\frac{\pi}{2}x)\sin(\pi y), \qquad (15)$$
$$u|_\Gamma(x,y) = \varphi(x,y) = 0,$$

where the analytical solution is:

$$u(x,y) = x\cos(\frac{\pi}{2})\sin(\pi y). \qquad (16)$$

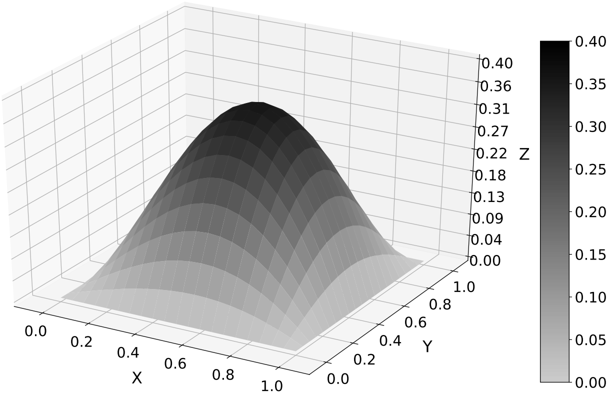

Figure 3. The exact solution for the equation (15)

Example 2 – the method errors estimates

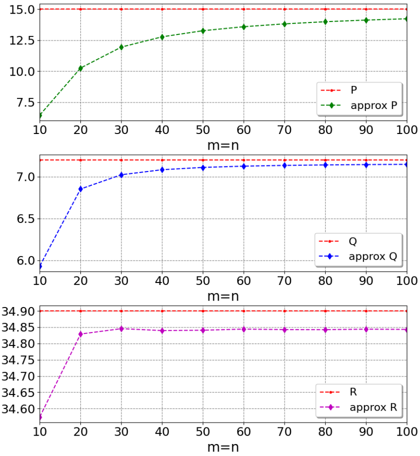

Figure 4. Estimates of the constants P, Q and R

Example 2 – numerical results

Table 2. Interval solutions and interval widths obtained in proper (PIA) and directed (DIA) interval arithmetic for the problem (15) at point $(x, y) = (0.5, 0.5)$. Parameters of the NM method: $GSL\_MC\_ITER = 25000$, $\epsilon = 10^{-8}$ and $\delta = 10^{-8}$. Exact solution $u(0.5, 0.5) \approx 0.353553390593273762$

$m = n$	$U(0.5, 0.5)$	Width
20 (PIA)	[0.34941684316762481, 0.35610061725233360]	0.006684
20 (DIA)	[0.34962483373204273, 0.35589262668791567]	0.006268
20 (NM)	[0.33915297428263464, 0.36240419420738459]	0.023252
40 (PIA)	[0.35237722718097839, 0.35434723485693209]	0.001971
40 (DIA)	[0.35258608866995258, 0.35413837336795790]	0.001553
40 (NM)	[0.34825826185867063, 0.35991260298977276]	0.011655
60 (PIA)	[0.35291693874340361, 0.35402163378308488]	0.001105
60 (DIA)	[0.35312596214058223, 0.35381261038590626]	0.000687
60 (NM)	[0.35083812253001074, 0.35861146659504341]	0.007774
80 (PIA)	[0.35310460462858216, 0.35390792521922393]	0.000804
80 (DIA)	[0.35331368474001861, 0.35369884510778749]	0.000386
80 (NM)	[0.35205135444988570, 0.35788372978196933]	0.005843
100 (PIA)	[0.35319115489527668, 0.35385542477058268]	0.000665
100 (DIA)	[0.35340026126546315, 0.35364631840039621]	0.000247
100 (NM)	[0.35276267037643749, 0.35742841430850891]	0.004665

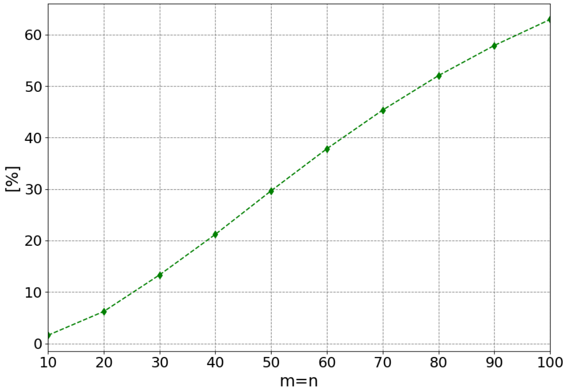

Figure 5. The difference in the widths of the result intervals obtained in the point $(x, y) = (0,5; 0,5)$ between the interval method of central differences in proper and directed floating point arithmetic (expressed as a percentage)

Summary of the results

Our research conducted so far has yielded the following results:

- implementations of the proper and directed interval arithmetic in Pascal and C++ language (including for types of arbitrary precision),
- second and higher order interval methods for Poisson's equation (in classical and generalized form),
- ways of estimating method errors experimentally.

There are several points to make improvements:

- ways of estimating method errors with mathematical proofs (to develop verified computing methods),
- the selection of optimal grid for the given problem,
- the use directed (or other types of) interval arithmetic.

Thank you for your attention!

References I

[1] Hoffmann, T., and Marciniak, A.
Solving the Poisson Equation by an Interval Difference Method of the Second Order.
Computational Methods in Science and Technology 19, 1 (2013), 13–21.

[2] Hoffmann, T., and Marciniak, A.
Solving the Generalized Poisson Equation in Proper and Directed Interval Arithmetic.
Computational Methods in Science and Technology 22, 4 (2016), 225–232.

[3] Jaulin, L., Kieffer, M., Didrit, O., and Walter, E.
Interval Analysis.
In *Applied Interval Analysis*. Springer, 2001, pp. 11–43.

[4] Marciniak, A.
Nakao's Method and An Interval Difference Scheme of Second Order for Solving The Elliptic BVP.
Computational Methods in Science and Technology 25, 2 (2019), 81–97.

[5] Marciniak, A., Jankowska, M. A., and Hoffmann, T.
An Interval Difference Method of Second Order for Solving an Elliptical BVP.
In *International Conference on Parallel Processing and Applied Mathematics* (2019), Springer, pp. 407–417.

[6] Markov, S.
On Directed Interval Arithmetic and Its Applications.
In *J. UCS The Journal of Universal Computer Science*. Springer, 1996, pp. 514–526.

[7] Markov, S., and Angelov, R.
An interval method for systems of ode.
In *International Symposium on Interval Mathematics* (1985), Springer, pp. 103–108.

References II

[8] Moore, R., Kearfott, R. B., and Cloud, M. J.
Introduction to Interval Analysis.
Society of Industrial and Applied Mathematics, Philadelphia, 2003.

[9] Moore, R. E.
A survey of interval methods for differential equations.
In *The 23rd IEEE Conference on Decision and Control* (1984), IEEE, pp. 1529–1535.

[10] Nakao, M. T.
A Numerical Approach to the Proof of Existence of Solutions for Elliptic Problems.
Japan Journal of Applied Mathematics 5, 2 (1988), 313.

[11] Nakao, M. T.
A Numerical Verification Method for The Existence of Weak Solutions for Nonlinear Boundary Value Problems.
Journal of Mathematical Analysis and Applications 164, 2 (1992), 489–507.

[12] Nakao, M. T., Plum, M., and Watanabe, Y.
Numerical Verification Methods and Computer-Assisted Proofs for Partial Differential Equations.
Springer Singapore, Singapore, 2019.

[13] Nedialkov, N. S., Jackson, K. R., and Pryce, J. D.
An effective high-order interval method for validating existence and uniqueness of the solution of an ivp for an ode.
Reliable Computing 7, 6 (2001), 449–465.

[14] Popova, E. D.
Extended Interval Arithmetic in IEEE Floating-Point Environment.
Interval Computations 4 (1994), 100–129.